

B9824 Foundations of Optimization

## Lecture 1: Introduction

Fall 2008

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### Outline

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1. Administrative matters
2. Introduction
3. Existence of optima
4. Local theory of unconstrained optimization

# Administrative Matters

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## Instructor

Prof. Ciamac Moallemi

Uris 416

email: [ciamac@gsb.columbia.edu](mailto:ciamac@gsb.columbia.edu)

## Office Hours

Drop by anytime M-F for quick questions, otherwise email for an appointment. (I will schedule formal office hours if there is demand.)

# Administrative Matters

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## GSB Calendar

No class on 10/17. Will reschedule.

## Coursework and Grading

- Homework (10%)
- Midterm (30%)
- Final (60%)

## Course Website

<http://angel.gsb.columbia.edu>

# Mathematical Prerequisites

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- Linear Algebra
  - Vectors and matrices over  $\mathbb{R}$
  - Null space, range
  - Transpose, inner product, norm
  - Eigenvalues of symmetric matrices; Spectral theorem
  - Positive definite and semidefinite matrices
- Calculus / Real Analysis
  - Open, closed, compact sets
  - Convergent sequences and subsequences
  - Continuity
  - Differentiability
  - Taylor series expansion
  - Mean value theorem
  - Implicit function theorem

## Background Reading

Bertsekas NLP, Appendix A.0-A.5; Boyd & Vandenberghe, Appendix A

# Required Texts

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- D. G. Luenberger, *Optimization by Vector Space Methods*. Wiley, 1969.
- D. P. Bertsekas, *Dynamic Programming and Optimal Control, Volume 1*, 3rd Edition. Athena Scientific, 2005.

Real Analysis:

- W. Rudin, *Principles of Mathematical Analysis*, 3rd Edition. McGraw-Hill, 1976.

Linear Algebra:

- G. Strang, *Linear Algebra and Its Applications*, 3rd Edition. Brooks Cole, 1988.

Optimization:

- D. P. Bertsekas, *Nonlinear Programming*, 2nd Edition. Athena Scientific, 1999.
- S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004. Available online at <http://www.stanford.edu/~boyd/cvxbook>.

## Introduction

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An optimization problem (program):

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

$x$  is the collection of **decision variables**

The real-valued function  $f(\cdot)$  is the **objective**

$\mathcal{C}$  is the **constraint set** (feasible set, search space), it is a subset of the domain of  $f$

$x^*$  is an **optimal solution** (global minimizer) if

$$f(x^*) \leq f(x), \quad \forall x \in \mathcal{C}$$

The **optimal value** is  $f(x^*)$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

- Maximization also falls within this framework

$$\text{maximize } f(x) \quad \Leftrightarrow \quad \text{minimize } -f(x)$$

- Problem classifications
  - continuous vs. discrete
  - deterministic vs. stochastic
  - static vs. dynamic

## Constraint Sets

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$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

The feasible set will usually be constructed as the **intersection** of a number of constraints, e.g.

$$\begin{aligned} \mathcal{C} = & \{x : h_1(x) = 0, \dots, h_m(x) = 0\} && \text{(equality constraints)} \\ & \cap \{x : g_1(x) \leq 0, \dots, g_r(x) \leq 0\} && \text{(inequality constraints)} \\ & \cap \Omega && \text{(set constraint)} \end{aligned}$$

## Motivating Example: Resource Allocation

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- Activities  $1, \dots, m$  (e.g., divisions of a firm)
- Resources  $1, \dots, n$  (e.g., capital, labor, etc.)
- Each activity consumes resources, and generates a benefit (utility, profit, etc.)
- Decision variables:

$x_{ij}$  = quantity of resource  $j$  allocated to activity  $i$

$$x_{ij} \geq 0$$

- The  $i$ th activity generates utility according to

$$U_i(x_{i1}, \dots, x_{in})$$

- The supply of the resources is limited, so we require that

$$\sum_{i=1}^m x_{ij} \leq C_j, \quad \forall j$$

## Motivating Example: Resource Allocation

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- Objective: maximize total utility

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^m U_i(x_{i1}, \dots, x_{in}) \\ &\text{subject to} && \sum_{i=1}^m x_{ij} \leq C_j, && \forall 1 \leq j \leq n \\ &&& x \geq 0, \\ &&& x \in \mathbb{R}^{m+n} \end{aligned}$$

## Motivating Example: Portfolio Optimization

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- Securities  $1, \dots, n$  are available for purchase
- Security  $i$  has return  $r_i$ , which is a random variable

$$E[r_i] = \mu_i, \quad \text{Cov}(r_i, r_j) = \Gamma_{ij}$$

- Decision variables:  $x_i =$  fraction of wealth to invest in security  $i$

$$x \geq 0, \quad \mathbf{1}^\top x = \sum_{i=1}^n x_i = 1$$

- Given a portfolio  $x$ ,

$$E[\text{return}] = \sum_{i=1}^n \mu_i x_i = \mu^\top x, \quad \text{Var}(\text{return}) = \sum_{i=1}^n \sum_{j=1}^n \Gamma_{ij} x_i x_j = x^\top \Gamma x$$

- The investor requires return  $\bar{\mu}$ , so

$$\mu^\top x = \bar{\mu}$$

## Motivating Example: Portfolio Optimization

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- Objective: minimize the variance of the portfolio return (risk)

$$\begin{aligned} &\text{minimize} && x^\top \Gamma x \\ &\text{subject to} && \mathbf{1}^\top x = 1, \\ & && \mu^\top x = \bar{\mu}, \\ & && x \geq 0, \\ & && x \in \mathbb{R}^n \end{aligned}$$

## Motivating Example: Production Planning

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- A manufacturer is planning the production goods  $1, \dots, n$  over the time horizon  $[0, T]$
- Goods are produced at the rate  $r(t) \in \mathbb{R}^n$  at time  $t$ ,  $r(t) \geq 0$
- $d(t) \in \mathbb{R}^n$  is the rate of demand at time  $t$ , manufacturer must meet this demand
- The inventory at time  $t$  is  $x(t) \in \mathbb{R}^n$
- Given a fixed initial inventory  $x(0)$ ,

$$x(t) = x(0) + \int_0^t [r(\tau) - d(\tau)] d\tau \geq 0$$

- Production cost rate  $c(r(t))$ , for some function  $c(\cdot) \geq 0$
- Holding/inventory cost rate  $h^\top x(t)$ , for some vector  $h \geq 0$  (linear)

## Motivating Example: Production Planning

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- Objective: minimize the total cost of the production plan

$$\text{minimize} \quad \int_0^T [c(r(t)) + h^\top x(t)] dt$$

$$\begin{aligned} \text{subject to} \quad & x(t) = x(0) + \int_0^t [r(\tau) - d(\tau)] d\tau \geq 0, \quad \forall t \in [0, T] \\ & r(t) \geq 0, \quad \forall t \in [0, T] \\ & r(\cdot) \in C([0, T], \mathbb{R}^n) \end{aligned}$$

Here,

$C([0, T], \mathbb{R}^n)$  = continuous functions from  $[0, T]$  to  $\mathbb{R}^n$

- Data analysis: fitting, statistical estimation
- Solution of equilibrium models
- Game theory
- Communications: scheduling, routing
- Computer-aided design

## The Basic Questions

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$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Does an optimal solution exist?

Can we characterize the set of optimal solutions? (Necessary & sufficient conditions)

Is the optimal solution unique?

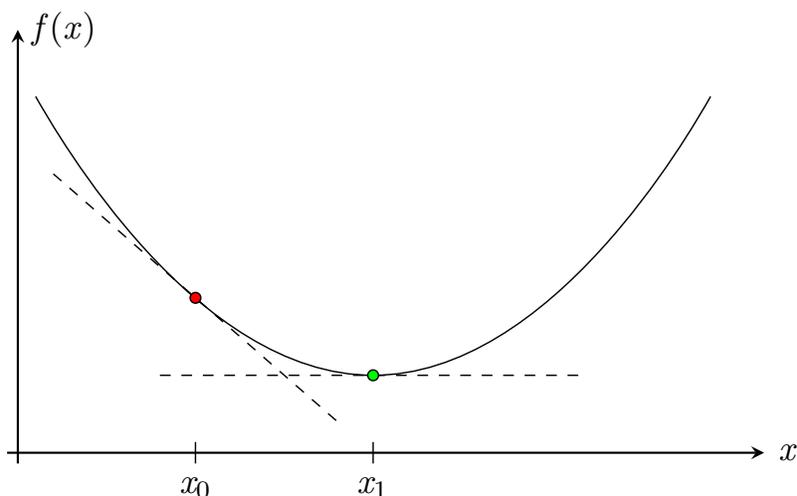
How sensitive is the solution to changes in the objective function or constraint set?

How can an optimal solution/the optimal value be efficiently computed?

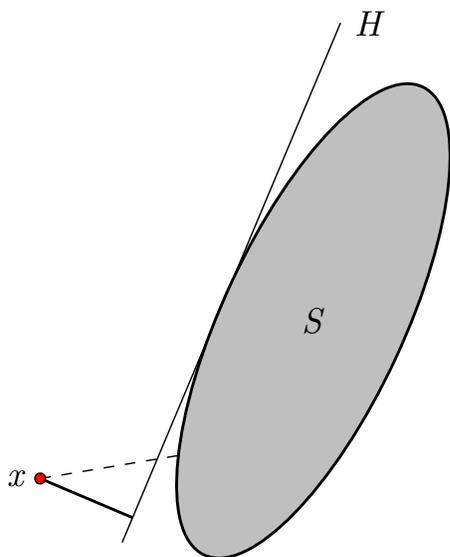
## Big Idea I: Differentials

The behavior of a “smooth” function  $f(\cdot)$  close to a point  $x$  can be approximated with derivatives, e.g., if  $y$  is close to  $x$ ,

$$f(y) \approx f(x) + f'(x)(y - x)$$



## Big Idea II: Duality



**Problem 1.** Find the smallest distance between a vector  $x$  and a point in the convex set  $S$

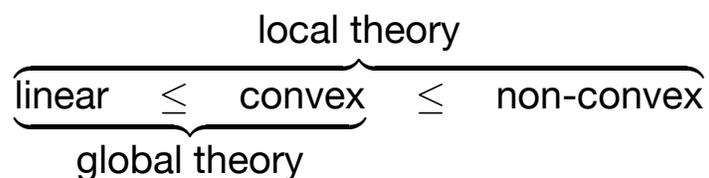
**Problem 2.** Find the largest distance between a vector  $x$  and a hyperplane  $H$  separating  $x$  from  $S$

These problems are equivalent!

minimize over points  $\Leftrightarrow$  maximize over hyperplanes

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && h_i(x) = 0, \quad \forall 1 \leq i \leq m \\ & && g_i(x) \leq 0, \quad \forall 1 \leq i \leq r \\ & && x \in \mathbb{R}^n \end{aligned}$$

- **Linear programming:**  $f(\cdot), \{h_i(\cdot)\}, \{g_i(\cdot)\}$  are linear
- **Convex programming:**  $f(\cdot), \{h_i(\cdot)\}, \{g_i(\cdot)\}$  are convex
- **Non-convex programming:** anything goes



## Rough Syllabus

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1. Introduction
2. Local theory of optimization (differentials, Lagrangians)
3. Global theory of optimization (convexity, duality)
4. Deterministic optimal control
5. Vector spaces: a unifying theory

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

When does an optimal solution exist?

**Example.**

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & x \in \mathbb{R} \end{array} \Rightarrow \text{Unbounded!}$$

**Example.**

$$\begin{array}{ll} \text{minimize} & 1/x \\ \text{subject to} & x > 0 \end{array} \Rightarrow \text{Bounded, but no optima!}$$

## Basic Topology

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**Definition.** An **open ball** (or, “neighborhood”) around a point  $x \in \mathbb{R}^n$  with radius  $r > 0$  is the set

$$N_r(x) \triangleq \{y \in \mathbb{R}^n : \|x - y\| < r\}$$

Here,

$$\|x\| \triangleq \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$$

Consider a set  $\mathcal{E} \subset \mathbb{R}^n$ .

**Definition.** A point  $x \in \mathcal{E}$  is an **interior point** if there exists an open ball  $N_r(x)$  such that  $N_r(x) \subset \mathcal{E}$ . The **interior**  $\text{int } \mathcal{E}$  is defined to be the set of all interior points of  $\mathcal{E}$ .

**Definition.**  $\mathcal{E}$  is **open** if  $\mathcal{E} = \text{int } \mathcal{E}$ .

**Definition.** A point  $x \in \mathbb{R}^n$  is a **closure point** of  $\mathcal{E}$  if, for every open ball  $N_r(x)$ , there exists  $y \in \mathcal{E}$  with  $y \in N_r(x)$ . The **closure**  $\text{cl } \mathcal{E}$  is defined to be the set of all closure points of  $\mathcal{E}$ .

**Definition.**  $\mathcal{E}$  is **closed** if every closure point if  $\mathcal{E} = \text{cl } \mathcal{E}$ .

## Convergence

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**Definition.** A sequence of vectors  $\{x_k\} \subset \mathbb{R}^n$  **converges** to a limit  $x \in \mathbb{R}^n$  if

$$\lim_{k \rightarrow \infty} \|x - x_k\| = 0$$

We say  $x_k \rightarrow x$ .

Consider a set  $\mathcal{E} \subset \mathbb{R}^n$ .

**Definition.**  $\mathcal{E}$  is **compact** if, given a sequence  $\{x_k\} \subset \mathcal{E}$ , there is a subsequence  $\{x_{k_i}\}$  converging to an element  $x \in \mathcal{E}$ .

**Definition.**  $\mathcal{E}$  is **bounded** if there exists a neighborhood  $N_r(x)$  such that  $\mathcal{E} \subset N_r(x)$ .

**Theorem.** (Heine-Borel) A set  $\mathcal{E} \subset \mathbb{R}^n$  is compact if and only if it is closed and bounded.

## Continuity

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Consider a real-valued function  $f(\cdot)$  defined on a domain  $\mathcal{X} \subset \mathbb{R}^n$ .

**Definition.**  $f(\cdot)$  is **continuous** at the point  $x \in \mathcal{X}$  if, for every sequence  $\{x_k\} \subset \mathcal{X}$  with  $x_k \rightarrow x$ ,

$$\lim_{k \rightarrow \infty} f(x_k) = f(x)$$

We say  $f(\cdot)$  is **continuous** if it is continuous at all points of  $\mathcal{X}$ .

Consider a set  $\mathcal{A} \subset \mathbb{R}$ , define the inverse image

$$f^{-1}(\mathcal{A}) \triangleq \{x \in \mathcal{X} : f(x) \in \mathcal{A}\}$$

**Theorem.** Assume that  $f(\cdot)$  is continuous. If the domain  $\mathcal{X}$  is open (respectively, closed) and  $\mathcal{A}$  is open (closed), then  $f^{-1}(\mathcal{A})$  is also open (closed).

## Existence Theorem I

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$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

When does an optimal solution exist?

**Theorem.** (Weierstrass) Assume that  $\mathcal{C}$  is non-empty and that  $f(\cdot)$  is continuous over  $\mathcal{C}$ .

If  $\mathcal{C}$  is compact, then the set of optimal solutions of  $f(\cdot)$  is non-empty and compact.

## Proof of Existence Theorem I

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Define

$$f^* = \inf_{x \in \mathcal{C}} f(x) \in \mathbb{R} \cup \{-\infty\}$$

Note:  $f^*$  always exists!

Given  $\gamma > f^*$ , the sub-level set

$$\mathcal{C}(\gamma) \triangleq \{x \in \mathcal{C} : f(x) \leq \gamma\}$$

must be non-empty and closed (continuity of  $f(\cdot)$ ). Then, since  $\mathcal{C}$  is compact,  $\mathcal{C}(\gamma)$  also compact. If  $\gamma_k \downarrow f^*$ , set of optimal solutions is

$$\mathcal{X}^* = \bigcap_{k=1}^{\infty} \mathcal{C}(\gamma_k)$$

The intersection of a collection of nested, non-empty compact sets is non-empty and compact.

$$\begin{aligned} & \text{minimize} && x^\top \Gamma x \\ & \text{subject to} && \mathbf{1}^\top x = 1, \\ & && \mu^\top x = \bar{\mu}, \\ & && x \geq 0, \\ & && x \in \mathbb{R}^n \end{aligned}$$

Check that:

- The objective is continuous  
⇒ true since it is a polynomial
- The feasible set is non-empty  
⇒ true iff  $\min_i \mu_i \leq \bar{\mu} \leq \max_i \mu_i$
- The feasible set is compact  
⇒ clearly bounded  
⇒ closed since it is the inverse image of a closed set under a continuous function

## Coerciveness

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**Example.** Consider the program

$$\begin{aligned} & \text{minimize} && x^2 \\ & \text{subject to} && x \in \mathbb{R} \end{aligned}$$

This has a unique optimal solution  $x^* = 0$ , but the Weierstrass Theorem does not apply since the feasible set is not compact.

Consider a real-valued function  $f(\cdot)$ .

**Definition.**  $f(\cdot)$  is **coercive** over a set  $\mathcal{C} \subset \mathbb{R}^n$ , if, for every sequence  $\{x_k\} \subset \mathcal{C}$  with  $\|x_k\| \rightarrow \infty$ ,

$$\lim_{k \rightarrow \infty} f(x_k) = \infty$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

When does an optimal solution exist?

**Theorem.** Assume that  $\mathcal{C}$  be non-empty and that  $f(\cdot)$  is continuous over  $\mathcal{C}$ .

If  $\mathcal{C}$  is closed and  $f(\cdot)$  is coercive over  $\mathcal{C}$ , then the set of optimal solutions of  $f(\cdot)$  is non-empty and compact.

**Proof.** Since  $f(\cdot)$  is coercive, that  $\mathcal{C}(\gamma)$  is non-empty and bounded for  $\gamma > f^*$ . Since  $\mathcal{C}$  is also closed, so is  $\mathcal{C}(\gamma)$ . Then,  $\mathcal{C}(\gamma)$  is compact. The proof proceeds as before.  $\square$

## Application: Unconstrained Quadratic Optimization

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Given a symmetric matrix  $\Gamma \in \mathbb{R}^{n \times n}$  and a vector  $b \in \mathbb{R}^n$ , consider:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top \Gamma x - b^\top x \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

What are sufficient conditions to guarantee the existence of an optimal solution?

**Answer:** If  $\lambda$  is the smallest eigenvalue of  $\Gamma$ , we have

$$\frac{1}{2}x^\top \Gamma x - b^\top x \geq \frac{\lambda}{2}\|x\|^2 - \|b\|\|x\|$$

Thus, if  $\lambda > 0$ , the objective is coercive. This is equivalent to  $\Gamma \succ 0$ , i.e.,  $\Gamma$  is positive definite.

The key condition for both theorems is that there exists some  $\gamma^*$  such that the sub-level set

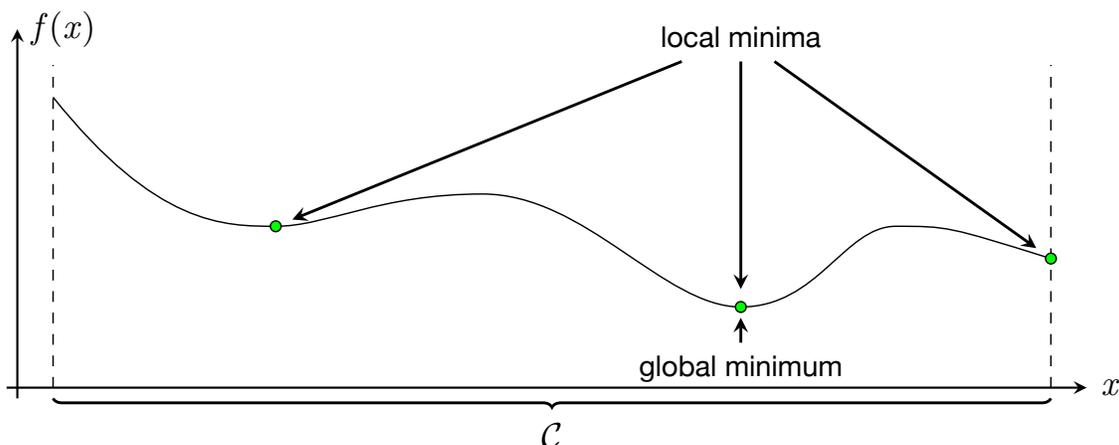
$$\mathcal{C}(\gamma^*) = \{x \in \mathcal{C} : f(x) \leq \gamma^*\}$$

is non-empty and compact.

Continuity of  $f(\cdot)$  was only used to establish that  $\mathcal{C}(\gamma)$  is closed for  $\gamma \leq \gamma^*$ . This is also implied by a weaker condition known as **lower semi-continuity**.

## Local Optimality

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

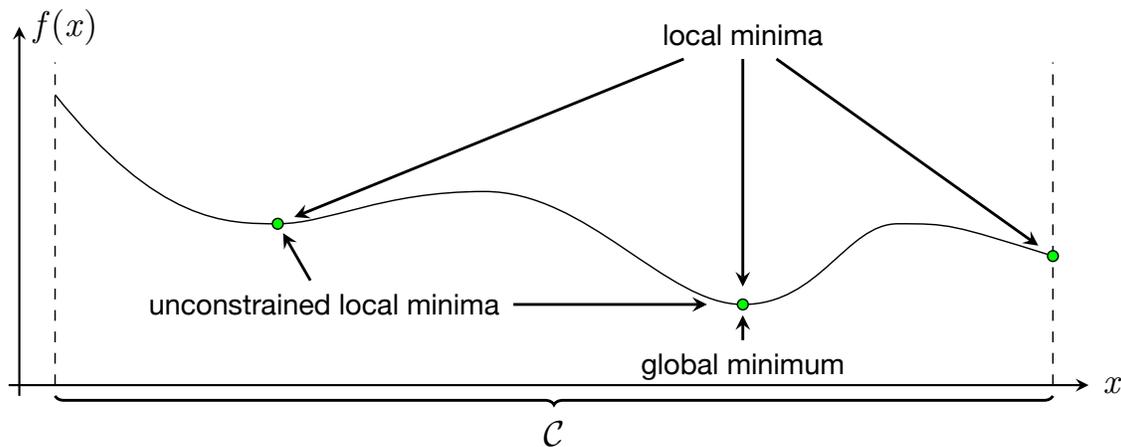


**Definition.** A point  $x \in \mathcal{C}$  is a **local minimum** if there exists a neighborhood  $N_r(x)$  such that

$$f(x) \leq f(y), \quad \forall y \in \mathcal{C} \cap N_r(x).$$

## Local Optimality

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$



**Definition.** A point  $x \in \mathcal{C}$  is an **unconstrained local minimum** if there exists a neighborhood  $N_r(x) \subset \mathcal{C}$  such that

$$f(x) \leq f(y), \quad \forall y \in N_r(x).$$

## Strict Local Optimality

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

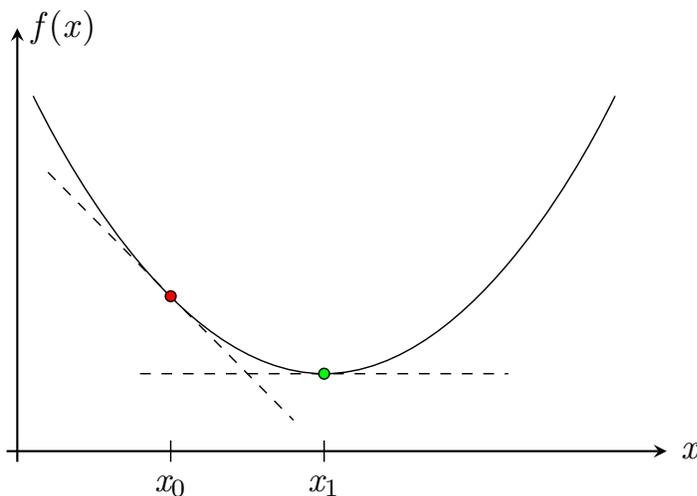
**Definition.** A point  $x \in \mathcal{C}$  is a **strict local minimum** if there exists a neighborhood  $N_r(x)$  such that

$$f(x) < f(y), \quad \forall y \in \mathcal{C} \cap N_r(x), \quad y \neq x.$$

**Definition.** A point  $x \in \mathcal{C}$  is an **strict unconstrained local minimum** if there exists a neighborhood  $N_r(x) \subset \mathcal{C}$  such that

$$f(x) < f(y), \quad \forall y \in N_r(x), \quad y \neq x.$$

$$f(x + h) \approx f(x) + f'(x)h + \frac{1}{2}f''(x)h^2$$



Necessary conditions:

$$f'(x) = 0$$

$$f''(x) \geq 0$$

Sufficient conditions:

$$f'(x) = 0$$

$$f''(x) > 0$$

## Differentiation

Consider a real-valued function  $f : \mathcal{X} \rightarrow \mathbb{R}$  with  $\mathcal{X} \subset \mathbb{R}^n$ .

**Definition.**  $f(\cdot)$  is **differentiable** at the point  $x \in \text{int } \mathcal{X}$  if there exists a vector  $\nabla f(x) \in \mathbb{R}^n$ , such that

$$\lim_{d \rightarrow 0} \frac{f(x + d) - f(x) - \nabla f(x)^\top d}{\|d\|} = 0.$$

$\nabla f(x)$  is known as the gradient.

**Definition.**  $f(\cdot)$  is **differentiable** over an open set  $\mathcal{U} \subset \mathcal{X}$  if it is differentiable at every point  $x \in \mathcal{U}$ . If, in addition, the components of the gradient  $\nabla f(x)$  are continuous over  $\mathcal{U}$ , we say  $f(\cdot)$  is **continuously differentiable** over  $\mathcal{U}$ .

**Note:** If  $f(\cdot)$  is differentiable at  $x$ , then

$$\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right),$$

where

$$\frac{\partial f(x)}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h}.$$

Here,  $e_i \in \mathbb{R}^n$  is the  $i$ th coordinate vector.

## The Hessian Matrix

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Consider a real-valued function  $f : \mathcal{X} \rightarrow \mathbb{R}$  with  $\mathcal{X} \subset \mathbb{R}^n$ .

**Definition.** Consider a point  $x \in \text{int } \mathcal{X}$ , and suppose that each component of the vector  $\nabla f(\cdot)$  is differentiable at  $x$ . We say that  $f(\cdot)$  is **twice differentiable** at  $x$ , and define the **Hessian** to be the matrix  $\nabla^2 f(x) \in \mathbb{R}^{n \times n}$  by

$$\nabla^2 f(x) = \left[ \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right]_{ij}$$

**Note:** If  $f(\cdot)$  is twice continuously differentiable in a neighborhood of  $x$ , then the Hessian is symmetric, i.e.,

$$\nabla^2 f(x) = \nabla^2 f(x)^\top$$

## Second Order Taylor Expansion

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Consider a real-valued function  $f : \mathcal{X} \rightarrow \mathbb{R}$  with  $\mathcal{X} \subset \mathbb{R}^n$ , and a point  $x \in \text{int } \mathcal{X}$ .

**Theorem.** Suppose that  $f(\cdot)$  is twice continuously differentiable over a neighborhood  $N_r(x)$ . Then, for all  $d \in N_r(0)$ ,

$$f(x + d) = f(x) + \nabla f(x)^\top d + \frac{1}{2} d^\top \nabla^2 f(x) d + o(\|d\|^2)$$

Formally, this means that if

$$R(d) = f(x + d) - \left( f(x) + \nabla f(x)^\top d + \frac{1}{2} d^\top \nabla^2 f(x) d \right),$$

then, for every  $C > 0$ , there exists a neighborhood  $N_\epsilon(0)$  and such that

$$|R(d)| < C\|d\|^2, \quad \forall d \in N_\epsilon(0)$$

## Necessary Conditions for Optimality

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$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

**Theorem.** Let  $x^* \in \text{int } \mathcal{C}$  be an unconstrained local minimum.

- (i) Suppose that  $f(\cdot)$  is continuously differentiable in a neighborhood of  $x^*$ . Then,

$$\nabla f(x^*) = 0 \quad (\text{first order necessary condition})$$

- (ii) If  $f(\cdot)$  is twice continuously differentiable in a neighborhood of  $x^*$ , then  $\nabla^2 f(x^*)$  is positive semidefinite, i.e.

$$\nabla^2 f(x^*) \succeq 0 \quad (\text{second order necessary condition})$$

## Necessary Conditions: Proof

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(i) Fix  $d \in \mathbb{R}^n \setminus \{0\}$ . Note that, by the definition of the gradient,

$$0 = \lim_{\alpha \rightarrow 0} \frac{f(x^* + \alpha d) - f(x^*) - \alpha \nabla f(x^*)^\top d}{\alpha \|d\|}$$

Thus,

$$\lim_{\alpha \rightarrow 0} \frac{f(x^* + \alpha d) - f(x^*)}{\alpha} = \nabla f(x^*)^\top d$$

For  $\alpha$  sufficiently small, however,

$$f(x^* + \alpha d) - f(x^*) \geq 0$$

Then,

$$\nabla f(x^*)^\top d \geq 0$$

Since  $d$  is arbitrary, it follows that  $\nabla f(x^*) = 0$ .

## Necessary Conditions: Proof

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(ii) Fix  $d \in \mathbb{R}^n$ . For  $\alpha$  sufficiently small, using a second order expansion,

$$\begin{aligned} f(x^* + \alpha d) - f(x^*) &= \alpha \nabla f(x^*)^\top d + \frac{1}{2} \alpha^2 d^\top \nabla^2 f(x^*) d + o(\alpha^2) \\ &= \frac{1}{2} \alpha^2 d^\top \nabla^2 f(x^*) d + o(\alpha^2) \end{aligned}$$

Then

$$0 \leq \frac{f(x^* + \alpha d) - f(x^*)}{\alpha^2} = \frac{1}{2} d^\top \nabla^2 f(x^*) d + \frac{o(\alpha^2)}{\alpha^2}$$

Taking the limit as  $\alpha \rightarrow 0$ , it follows that

$$0 \leq d^\top \nabla^2 f(x^*) d$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

**Theorem.** Consider a point  $x^* \in \text{int } \mathcal{C}$ . Suppose that  $f(\cdot)$  is twice continuously differentiable in a neighborhood  $N_r(x^*) \subset \mathcal{C}$  of  $x^*$ , and that

$$\nabla f(x^*) = 0, \quad \nabla^2 f(x^*) \succ 0.$$

Then,  $x^*$  is a **strict** unconstrained local minimum.

## Sufficient Conditions: Proof

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Let  $\lambda > 0$  be the smallest eigenvalue of  $\nabla^2 f(x^*)$ .

Fix  $d \in N_r(0) \setminus \{0\}$ ,

$$\begin{aligned} f(x^* + d) - f(x^*) &= \nabla f(x^*)^\top d + \frac{1}{2} d^\top \nabla^2 f(x^*) d + o(\|d\|^2) \\ &\geq \frac{\lambda}{2} \|d\|^2 + o(\|d\|^2) \\ &= \left( \frac{\lambda}{2} + \frac{o(\|d\|^2)}{\|d\|^2} \right) \|d\|^2 \end{aligned}$$

Pick  $0 < \epsilon \leq r$  and  $\gamma > 0$  so that

$$\frac{\lambda}{2} + \frac{o(\|d\|^2)}{\|d\|^2} \geq \frac{\gamma}{2}, \quad \forall d \text{ with } 0 < \|d\| < \epsilon$$

Then, for all  $d \in N_\epsilon(0) \setminus \{0\}$ ,

$$f(x^* + d) \geq f(x^*) + \frac{\gamma}{2} \|d\|^2 > f(x^*)$$

## Application of Necessary Conditions

---

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

How can we find the optimal solution(s)?

(Assuming  $f(\cdot)$  is continuously differentiable on  $\text{int } \mathcal{C}$ )

- (i) Check that a global minima exists (e.g., Weierstrass' Theorem, coerciveness)
- (ii) Find a set of possible unconstrained local minima via the necessary condition

$$\nabla f(x) = 0$$

- (iii) Find the set of “boundary” points  $\mathcal{C} \setminus \text{int } \mathcal{C}$
- (iv) The global minima must be among the points in (ii) and (iii), so evaluate  $f(\cdot)$  at each of these points and find those with the smallest value

## Application of Necessary Conditions

---

**Example.**  $f : [a, b] \rightarrow \mathbb{R}$  continuous on  $[a, b]$  and continuously differentiable on  $(a, b)$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Global minima must exist and are contained in the set

$$\{a, b\} \cup \{x \in (a, b) : f'(x) = 0\}$$

**Example.**  $\Gamma \in \mathbb{R}^{n \times n}$ ,  $\Gamma \succ 0$ ,  $b \in \mathbb{R}^n$

$$\begin{array}{ll} \text{minimize} & x^\top \Gamma x - b^\top x \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

Global minima must exist, and  $\mathcal{C} \setminus \text{int } \mathcal{C}$  is empty, so global minima must be unconstrained local minima. First order necessary conditions:

$$\Gamma x - b = 0 \quad \Rightarrow \quad x^* = \Gamma^{-1} b$$

The necessary conditions for unconstrained local optima are only useful if the boundary  $\mathcal{C} \setminus \text{int } \mathcal{C}$  is empty or “small”, or if the (for other reasons) we know the global minima will not occur on the boundary.

**Example.** (Portfolio optimization)

$$\begin{array}{ll} \text{minimize} & x^\top \Gamma x \\ \text{subject to} & \mathbf{1}^\top x = 1, \\ & \mu^\top x = \bar{\mu}, \\ & x \geq 0, \\ & x \in \mathbb{R}^n \end{array}$$

The interior of the constraint set is empty, no unconstrained local optima!

## Sensitivity Analysis

---

Consider a function  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ , and, for a fixed  $a \in \mathbb{R}^m$ , the unconstrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x, a) \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

Denote by  $x^*(a)$  a local minimizer, assuming it exists, and define  $f^*(a) \triangleq f(x^*(a), a)$ .

$a$  is a **parameter vector**. We would like to understand how the local minimum  $x^*(a)$  and the associated value  $f^*(a)$  are sensitive to changes in  $a$ .

## Differentiation of Vector-Valued Functions

---

Consider a vector-valued function  $F : \mathcal{X} \rightarrow \mathbb{R}^m$ ,  $\mathcal{X} \subset \mathbb{R}^n$ , and a point  $x \in \text{int } \mathcal{X}$ . We can analyze  $F(\cdot)$  in terms of component functions  $F_i : \mathcal{X} \rightarrow \mathbb{R}$  by

$$F(x) = (F_1(x), \dots, F_m(x)).$$

**Definition.**  $F(\cdot)$  is **differentiable** at the point  $x$  each component function  $F_i(\cdot)$  is differentiable at  $x$ . We define the **gradient** to be the matrix  $\nabla F(x) \in \mathbb{R}^{n \times m}$  with

$$\nabla F(x) = [\nabla F_1(x), \dots, \nabla F_m(x)]$$

If  $F(\cdot)$  is differentiable at  $x$ , then

$$\nabla F(x)_{ij} = \frac{\partial F_j(x)}{\partial x_i}$$

## Differentiation of Vector-Valued Functions

---

**Theorem.** (Chain Rule) If  $f : \mathcal{X} \rightarrow \mathbb{R}^m$ ,  $\mathcal{X} \subset \mathbb{R}^n$ , is differentiable at  $x \in \text{int } \mathcal{X}$  and  $g : \mathcal{Y} \rightarrow \mathbb{R}^p$ ,  $\mathcal{Y} \subset \mathbb{R}^m$ , is differentiable at  $f(x) \in \text{int } \mathcal{Y}$ , then the composition

$$h(x) = g(f(x))$$

is differentiable at  $x$ , and

$$\nabla h(x) = \nabla f(x) \nabla g(f(x))$$

$$\begin{array}{ll} \text{minimize} & f(x, a) & x^*(a) = \text{local minimum} \\ \text{subject to} & x \in \mathbb{R}^n & f^*(a) = f(x^*(a), a) \end{array}$$

First order conditions:  $\nabla_x f(x^*(a), a) = 0$

Taking derivatives:  $\nabla x^*(a) \nabla_{xx}^2 f(x^*(a), a) + \nabla_{xa}^2 f(x^*(a), a) = 0$

Sensitivity of local minimum:

$$\nabla x^*(a) = -\nabla_{xa}^2 f(x^*(a), a) \left( \nabla_{xx}^2 f(x^*(a), a) \right)^{-1}$$

Sensitivity of value:

$$\begin{aligned} \nabla f^*(a) &= \nabla x^*(a) \nabla_x f(x^*(a), a) + \nabla_a f(x^*(a), a) \\ &= \nabla_a f(x^*(a), a) \end{aligned}$$

## Implicit Function Theorem

---

**Theorem.** Let  $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a function with

- (i)  $F(\bar{x}, \bar{y}) = 0$
- (ii)  $F(\cdot, \cdot)$  is continuous and has a continuous and invertible gradient matrix  $\nabla_x F(x, y)$  in an open set containing  $(\bar{x}, \bar{y})$

Then, there exist open sets  $\mathcal{U}_x \subset \mathbb{R}^n$  and  $\mathcal{U}_y \subset \mathbb{R}^m$  with  $x \in \mathcal{U}_x$  and  $y \in \mathcal{U}_y$  and a continuous function  $\phi : \mathcal{U}_y \rightarrow \mathcal{U}_x$  such that

- (i)  $\bar{x} = \phi(\bar{y})$  and  $F(\phi(y), y) = 0$  for all  $y \in \mathcal{U}_y$
- (ii) The function  $\phi(\cdot)$  is unique in the sense that, if  $x \in \mathcal{U}_x$ ,  $y \in \mathcal{U}_y$ , and  $F(x, y) = 0$ , then  $x = \phi(y)$
- (iii) If  $F(\cdot, \cdot)$  is  $k$  times continuously differentiable the same is true for  $\phi(\cdot)$ , and

$$\nabla \phi(y) = -\nabla_y F(\phi(y), y) (\nabla_x F(\phi(y), y))^{-1}, \quad \forall y \in \mathcal{U}_y$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

Two types of analysis:

- **Analytical.** Use necessary and sufficient conditions to analytically solve for optimal solutions, or establish structural properties of optimal solutions.
- **Numerical.** Use necessary and sufficient conditions to numerically solve for optimal solutions.

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

Suppose  $f(\cdot)$  is differentiable at  $x \in \mathbb{R}^n$ , and  $d \in \mathbb{R}^n$  is a vector with  $\nabla f(x)^\top d < 0$ . Then,

$$\lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha} = \nabla f(x)^\top d < 0$$

Thus,  $d$  is a **descent direction** and there exists  $\epsilon > 0$  such that for all  $0 < \alpha \leq \epsilon$ ,

$$f(x + \alpha d) < f(x)$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

- Pick starting point  $x^{(0)} \in \mathbb{R}^n$
- For each  $k \geq 0$ , pick a descent direction  $d^{(k)} \in \mathbb{R}^n$  with  $\nabla f(x^{(k)})^\top d^{(k)} < 0$  and a step-size  $\alpha_k > 0$ , set

$$x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)}$$

- Stop according to a termination criteria, for example when

$$\|\nabla f(x^{(k)})\| < \delta,$$

for some tolerance  $\delta > 0$ .

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

- Steepest descent

$$x^{(k+1)} = x^{(k)} - \alpha_k \nabla f(x^{(k)})$$

- Newton's method

$$x^{(k+1)} = x^{(k)} - \alpha_k \left( \nabla^2 f(x^{(k)}) \right)^{-1} \nabla f(x^{(k)})$$

B9824 Foundations of Optimization

## Lecture 2: Local Theory of Optimization I

Fall 2008

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### Outline

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1. Constrained local optimality
2. Necessary conditions for equality constraints
3. Lagrangian cookbook recipe
4. Sufficient conditions for equality constraints
5. Sensitivity analysis

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

**Definition.** A point  $x \in \mathcal{C}$  is a **local minimum** if there exists a neighborhood  $N_r(x)$  such that

$$f(x) \leq f(y), \quad \forall y \in \mathcal{C} \cap N_r(x).$$

We are interested in characterizing local minima that are not in **int**  $\mathcal{C}$ . The constraint set plays an fundamental role in this case.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

If  $f(\cdot)$  is differentiable at a point  $x^* \in \mathcal{C}$ , and  $d \in \mathbb{R}^n$ , then

$$\lim_{\alpha \downarrow 0} \frac{f(x^* + \alpha d) - f(x^*)}{\alpha} = \nabla f(x^*)^\top d$$

If  $x^*$  is a local minimum and  $d$  is a “feasible” direction, then

$$0 \leq \nabla f(x^*)^\top d$$

## The Descent and Tangent Cones

---

minimize  $f(x)$   $x^* \in \mathcal{C}$ ,  $f(\cdot)$  continuously differentiable in  
subject to  $x \in \mathcal{C} \subset \mathbb{R}^n$  a neighborhood of  $x^*$

**Definition.** The set of **descent directions** of the objective function  $f(\cdot)$  at  $x^*$  is the set

$$\mathcal{D}(x^*) = \{d \in \mathbb{R}^n : \nabla f(x^*)^\top d < 0\}$$

**Definition.** The **tangent cone**  $\mathcal{T}(x^*)$  of the constraint set  $\mathcal{C}$  at  $x^*$  is the set of directions  $d \in \mathbb{R}^n$  such that either

- (i)  $d = 0$
- (ii) there exists a sequence  $\{x_k\} \subset \mathcal{C}$  with  $x_k \rightarrow x^*$  and

$$\frac{x_k - x^*}{\|x_k - x^*\|} \rightarrow \frac{d}{\|d\|}$$

## Local Optimality Necessary Condition

---

minimize  $f(x)$   $x^* \in \mathcal{C}$ ,  $f(\cdot)$  continuously differentiable in  
subject to  $x \in \mathcal{C} \subset \mathbb{R}^n$  a neighborhood of  $x^*$

**Theorem.** If  $x^*$  is a local minimum, then there is no descent direction in the tangent cone. That is,

$$\mathcal{D}(x^*) \cap \mathcal{T}(x^*) = \emptyset$$

## Local Optimality Necessary Condition: Proof

---

Consider  $d \in \mathcal{T}(x^*) \setminus \{0\}$ . There exists  $\{x_k\} \subset \mathcal{C}$ ,  $x_k \neq x^*$ ,  $x_k \rightarrow x^*$  such that

$$\frac{x_k - x^*}{\|x_k - x^*\|} \rightarrow \frac{d}{\|d\|}$$

Define

$$\zeta_k \triangleq \frac{x_k - x^*}{\|x_k - x^*\|} - \frac{d}{\|d\|}, \quad d_k \triangleq d + \|d\|\zeta_k$$

Then,  $\zeta_k \rightarrow 0$  and  $d_k \rightarrow d$ .

By the mean value theorem,

$$f(x_k) = f(x^*) + \nabla f(\tilde{x}_k)^\top (x_k - x^*)$$

where  $\tilde{x}_k$  is a point on the line segment between  $x_k$  and  $x^*$ .

## Local Optimality Necessary Condition: Proof

---

Equivalently,

$$f(x_k) = f(x^*) + \frac{\|x_k - x^*\|}{\|d\|} \nabla f(\tilde{x}_k)^\top d_k$$

If  $d \in \mathcal{D}(x^*)$ , then  $\nabla f(x^*)^\top d < 0$ . Thus, for  $k$  sufficiently large,  $\nabla f(\tilde{x}_k)^\top d_k < 0$ . Then,

$$f(x_k) < f(x^*),$$

which contradicts local minimality of  $x^*$ .

# Equality Constrained Optimization

---

Consider

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && h_i(x) = 0, \dots, h_m(x) = 0, \\ & && x \in \mathbb{R}^n \end{aligned}$$

where

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad h_i : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \forall 1 \leq i \leq m$$

- Assume that  $f(\cdot)$  and  $\{h_i(\cdot)\}$  are continuously differentiable on  $\mathbb{R}^n$
- The necessary and sufficient conditions are also true if these functions are just defined and continuously differentiable in a neighborhood of the local minimum

# Equality Constrained Optimization

---

$$\begin{aligned} f : \mathbb{R}^n &\rightarrow \mathbb{R} & \text{minimize} & f(x) \\ h : \mathbb{R}^n &\rightarrow \mathbb{R}^m & \text{subject to} & h(x) = 0, \\ & & & x \in \mathbb{R}^n \end{aligned}$$

Assume that  $x^*$  is a feasible point and  $d \in \mathbb{R}^n$  is a direction. For small  $\alpha > 0$ ,

$$h(x^* + \alpha d) \approx h(x^*) + \nabla h(x^*)^\top (\alpha d) = \alpha \nabla h(x^*)^\top d$$

**Definition.** The cone of **first order feasible variations** at a point  $x^* \in \mathbb{R}^n$  is the set

$$\mathcal{V}(x^*) = \{d \in \mathbb{R}^n : \nabla h(x^*)^\top d = 0\}$$

**Note:** If  $d \in \mathcal{V}(x^*)$ , then  $-d \in \mathcal{V}(x^*)$ .

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ & x \in \mathbb{R}^n \end{array}$$

**Definition.** A point  $x^* \in \mathbb{R}^n$  is a **regular point** if it is feasible and if the constraint gradients

$$\nabla h_1(x^*), \dots, \nabla h_m(x^*)$$

are linearly independent.

**Note:** If  $m > n$ , no regular points exist.

If  $m = 1$ , regularity is equivalent to  $\nabla h_1(x^*) \neq 0$ .

## Regularity Lemma

---

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ & x \in \mathbb{R}^n \end{array}$$

**Lemma.** Let  $x^*$  be a regular point. Then,

- (i) For each  $d \in \mathcal{V}(x^*)$ , there exists  $\tau > 0$  and a curve  $x : (-\tau, \tau) \rightarrow \mathbb{R}^n$  such that
  - (a)  $x(0) = x^*$ ,  $h(x(t)) = 0$  for  $t \in (-\tau, \tau)$
  - (b)  $x(\cdot)$  is continuously differentiable, and  $\dot{x}(0) = d$
  - (c) if  $h(\cdot)$  is twice continuously differentiable, then so is  $x(\cdot)$
- (ii)  $\mathcal{T}(x^*) = \mathcal{V}(x^*)$

## Regularity Lemma: Proof

---

(i) Fix  $d \in \mathcal{V}(x^*)$ . Given a scalar  $t$ , consider the  $m$  equations

$$h(x^* + td + \nabla h(x^*)u(t)) = 0$$

for an unknown  $u(t) \in \mathbb{R}^m$ . For  $t = 0$ , this has solution  $u(0) = 0$ . The gradient w.r.t.  $u$  is at  $t = 0$ ,  $u = 0$  is

$$\nabla h(x^*)^\top \nabla h(x^*)$$

This is invertible since the columns of  $\nabla h(x^*)$  are linearly independent. By the implicit function theorem, for some  $\tau > 0$ , a solution  $u(t)$  exists for  $t \in (-\tau, \tau)$ .

Define  $x(t) \triangleq x^* + td + \nabla h(x^*)u(t)$ . Differentiating  $h(x(t)) = 0$  at with respect to  $u$  at  $t = 0$ ,

$$0 = d^\top \nabla h(x^*) + \dot{u}(0)^\top \nabla h(x^*)^\top \nabla h(x^*)$$

Since  $d \in \mathcal{V}(x^*)$ ,  $\dot{u}(0) = 0$  thus  $\dot{x}(0) = d$ .

## Regularity Lemma: Proof

---

(ii)  $\mathcal{V}(x^*) \subset \mathcal{T}(x^*)$ : If  $d \in \mathcal{V}(x^*) \setminus \{0\}$ , define  $x(t)$  from (i), and there exists a sequence  $t_k \subset (0, \tau)$ ,  $t_k \rightarrow 0$ , with  $x_k \triangleq x(t_k) \neq x^*$ . Then,

$$\frac{x_k - x^*}{\|x_k - x^*\|} \rightarrow \frac{\dot{x}(0)}{\|\dot{x}(0)\|} = \frac{d}{\|d\|}$$

by the mean value theorem applied to  $x(t)$ .

$\mathcal{T}(x^*) \subset \mathcal{V}(x^*)$ : Consider  $d \in \mathcal{T}(x^*) \setminus \{0\}$ , and an associated  $\{x_k\}$ . By the mean value theorem,

$$0 = h(x_k) = h(x^*) + \nabla h(\tilde{x}_k)^\top (x_k - x^*)$$

Thus,

$$\nabla h(\tilde{x}_k)^\top \frac{x_k - x^*}{\|x_k - x^*\|} = 0$$

Take the limit as  $k \rightarrow \infty$ .

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ & x \in \mathbb{R}^n \end{array}$$

**Theorem.** If  $x^*$  is a local minimum that is a regular point, then

$$\nabla f(x^*)^\top d = 0$$

for all directions  $d \in \mathcal{V}(x^*)$ . In particular, there is no descent direction that is a first order feasible variation.

**Proof.** Since  $\mathcal{T}(x^*) = \mathcal{V}(x^*)$ , we have  $\mathcal{D}(x^*) \cap \mathcal{V}(x^*) = \emptyset$ . Assume  $d \in \mathcal{V}(x^*)$ . Then  $\nabla f(x^*)^\top d \geq 0$ . Since  $-d \in \mathcal{V}(x^*)$ , the result follows. □

## A Linear Algebra Lemma

---

**Definition.** Consider a matrix  $A \in \mathbb{R}^{m \times n}$ . The **kernel** (nullspace) is the set

$$\ker A \triangleq \{x \in \mathbb{R}^n : Ax = 0\}$$

The **image** (range) is the set

$$\text{im } A \triangleq \{y \in \mathbb{R}^m : y = Ax, x \in \mathbb{R}^n\}$$

**Definition.** Given a set  $\mathcal{S} \subset \mathbb{R}^n$ , the **orthogonal complement** is defined to be the set

$$\mathcal{S}^\perp \triangleq \{x \in \mathbb{R}^n : x^\top y = 0, \forall y \in \mathcal{S}\}$$

**Lemma.** If  $A \in \mathbb{R}^{m \times n}$  is a matrix, then  $\text{im } A = [\ker(A^\top)]^\perp$ .

In other words, given  $z \in \mathbb{R}^m$ ,

$$z = Ax \text{ for some } x \in \mathbb{R}^n \iff z^\top y = 0 \text{ for all } y \text{ with } A^\top y = 0$$

## Linear Algebra Lemma: Proof

---

Note that if  $\mathcal{S} \subset \mathbb{R}^k$  is a subspace, then  $(\mathcal{S}^\perp)^\perp = \mathcal{S}$ . So, we will prove  $\ker(A^\top) = [\text{im } A]^\perp$ .

$\ker(A^\top) \subset [\text{im } A]^\perp$ :

If  $z \in \ker(A^\top)$  and  $y = Ax$  for some  $x \in \mathbb{R}^n$ ,

$$z^\top y = z^\top Ax = 0 \quad \Rightarrow \quad z \in [\text{im } A]^\perp$$

$[\text{im } A]^\perp \subset \ker(A^\top)$ :

If  $z \in [\text{im } A]^\perp$ , then

$$z^\top Ax = 0, \quad \forall x \in \mathbb{R}^n \quad \Rightarrow \quad (A^\top z)^\top x = 0, \quad \forall x \in \mathbb{R}^n \quad \Rightarrow \quad A^\top z = 0$$

## Linear Algebra Lemma & Local Optimality

---

Our local optimality necessary condition was, for a regular point  $x^*$ ,

$$\nabla f(x^*)^\top d = 0, \quad \forall d \in \mathcal{V}(x^*)$$

In other words,

$$\nabla f(x^*) \in \mathcal{V}(x^*)^\perp = [\ker \nabla h(x^*)^\top]^\perp \quad \Rightarrow \quad \nabla f(x^*) \in \text{im } \nabla h(x^*)$$

Equivalently, there exists  $\lambda \in \mathbb{R}^m$  such that

$$\nabla f(x^*) + \nabla h(x^*)\lambda = 0$$

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ & x \in \mathbb{R}^n \end{array}$$

**Theorem.** If  $x^*$  is a local minimum that is a regular point, then there exists a unique vector  $\lambda^* \in \mathbb{R}^m$  called a **Lagrange multiplier**, such that

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla h_i(x^*) = 0.$$

If, in addition,  $f(\cdot)$  and  $h(\cdot)$  are twice continuously differentiable,

$$d^\top \left( \nabla^2 f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla^2 h_i(x^*) \right) d \geq 0, \quad \forall d \in \mathcal{V}(x^*).$$

## Necessary Conditions: Proof

First order conditions: existence of  $\lambda^*$  follows from earlier discussion,  $\lambda^*$  is unique since the columns of  $\nabla h(x^*)$  are linearly independent.

Second order conditions: consider  $d \in \mathcal{V}(x^*)$ . Define the path  $x(t)$  by the regularity lemma with  $h(x(t)) = 0$ ,  $x(0) = x^*$ ,  $\dot{x}(0) = d$ . If  $g(t) \triangleq f(x(t))$ ,  $t = 0$  must be an unconstrained local minimum of  $g(t)$ . Then,

$$\begin{aligned} 0 \leq \ddot{g}(0) &= \dot{x}(0)^\top \nabla^2 f(x^*) \dot{x}(0) + \ddot{x}(0)^\top \nabla f(x^*) \\ &= d^\top \nabla^2 f(x^*) d + \ddot{x}(0)^\top \nabla f(x^*) \end{aligned}$$

Differentiate  $\ell(t) \triangleq \lambda^\top h(x(t)) = 0$  twice at  $t = 0$  to obtain

$$0 = \ddot{\ell}(0) = d^\top \left( \sum_{i=1}^m \lambda_i \nabla^2 h_i(x^*) \right) d + \ddot{x}(0)^\top \nabla h(x^*) \lambda$$

**Definition.** The **Lagrangian** function  $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  is defined by

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) = f(x) + \lambda^\top h(x)$$

The necessary conditions can be written as

$$\begin{aligned} \nabla_x L(x^*, \lambda^*) &= 0, & \nabla_\lambda L(x^*, \lambda^*) &= 0, \\ d^\top \nabla_{xx}^2 L(x^*, \lambda^*) d &\geq 0, & \forall d \in \mathcal{V}(x^*). \end{aligned}$$

- Geometric interpretation
- Penalty function interpretation

## The Lagrangian Cookbook Recipe

---

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ & x \in \mathbb{R}^n \end{array}$$

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i h_i(x)$$

- (i) Check that a global minima exists
- (ii) Find the set of  $(x^*, \lambda^*)$  satisfying the necessary conditions

$$\nabla_x L(x^*, \lambda^*) = 0, \quad \nabla_\lambda L(x^*, \lambda^*) = 0$$

- (iii) Find the set of non-regular points
- (iv) The global minima must be among the points in (ii) and (iii)

(Assuming  $f(\cdot)$ ,  $h(\cdot)$  continuously differentiable on  $\mathbb{R}^n$ )

## Example

$$\begin{aligned} &\text{minimize} && f(x) = x_1 + x_2 \\ &\text{subject to} && h_1(x) = x_1^2 + x_2^2 - 2 = 0, \\ &&& x \in \mathbb{R}^3 \end{aligned}$$

Objective & constraints continuously differentiable

Global minima exist (Weierstrass)

First order conditions:

$$\begin{aligned} x_1^* + 2\lambda^* &= 0 \\ x_2^* + 2\lambda^* &= 0 \\ (x_1^*)^2 + (x_2^*)^2 &= 2 \end{aligned} \quad (x_1^*, x_2^*, \lambda^*) = \begin{cases} (-1, -1, \frac{1}{2}) \\ (1, 1, -\frac{1}{2}) \end{cases}$$

Regularity:  $\nabla h_1(x) = (2x_1, 2x_2) \neq 0$  Global minimum:  $x^* = (-1, -1)$

## Example: Maximum Volume Box

$$\begin{aligned} &\text{maximize} && f(x) = x_1 x_2 x_3 \\ &\text{subject to} && h_1(x) = x_1 x_2 + x_2 x_3 + x_1 x_3 - c/2 = 0, \quad (c > 0) \\ &&& x \geq 0, x \in \mathbb{R}^3 \end{aligned}$$

Objective & equality constraints continuously differentiable

Global maxima exist (why?), must have  $x^* > 0$

First order conditions: (when  $x^* > 0$ )

$$\begin{aligned} x_2^* x_3^* + \lambda^* (x_2^* + x_3^*) &= 0 \\ x_1^* x_3^* + \lambda^* (x_1^* + x_3^*) &= 0 \\ x_1^* x_2^* + \lambda^* (x_1^* + x_2^*) &= 0 \\ x_1^* x_2^* + x_2^* x_3^* + x_1^* x_3^* &= c/2 \end{aligned} \quad \begin{aligned} x_1^* &= x_2^* = x_3^* = \sqrt{c/6} \\ \lambda^* &= -\frac{1}{2}\sqrt{c/6} \end{aligned}$$

Regularity:  $\nabla h_1(x) = (x_2 + x_3, x_1 + x_3, x_1 + x_2) \neq 0$  if  $x > 0$

Unique global maximum:  $x_1^* = x_2^* = x_3^* = \sqrt{c/6}$

## Example

$$\begin{aligned} &\text{minimize} && f(x) = x_1 + x_2 \\ &\text{subject to} && h_1(x) = (x_1 - 1)^2 + x_2^2 - 1 = 0 \\ &&& h_2(x) = (x_1 - 2)^2 + x_2^2 - 4 = 0 \\ &&& x \in \mathbb{R}^2 \end{aligned}$$

$x^* = (0, 0)$  is the only feasible point, thus global minimum

First order conditions:

$$1 + 2\lambda_1^*(x_1^* - 1) + 2\lambda_2^*(x_1^* - 2) = 0$$

$$1 + 2\lambda_1^*x_2^* + 2\lambda_2^*x_2^* = 0$$

$$(x_1^* - 1)^2 + (x_2^*)^2 = 1$$

$$(x_1^* - 2)^2 + (x_2^*)^2 = 4$$

No solution to necessary conditions!

Regularity:

$$\nabla h_1(x) = (2x_1 - 2, 2x_2)$$

$$\nabla h_2(x) = (2x_1 - 4, 2x_2)$$

$x^* = (0, 0)$  is not regular!

## Constraint Qualification

Let  $x^*$  be a local minimum.

**Constraint qualification** refers to conditions on the constraints that guarantee the existence of Lagrange multipliers satisfying necessary conditions at  $x^*$ . Examples:

- Regularity
- Linear constraints [homework!]

More generally:

- $\mathcal{D}(x^*) \cap \mathcal{T}(x^*) = \emptyset$  (since  $x^*$  is a local minimum)
- $\mathcal{D}(x^*) \cap \mathcal{V}(x^*) = \emptyset$  implies the existence of Lagrange multipliers
- Quasiregularity:  $\mathcal{V}(x^*) = \mathcal{T}(x^*)$

**Note:** Regularity is a property of the **representation** of constraints, not the constraint set.

**Example.**

$h(x) = x_1 \Rightarrow \mathcal{C} = \{x \in \mathbb{R}^2 : h(x) = 0\} = \{(x_1, x_2) : x_1 = 0\}$   
All points in  $\mathcal{C}$  are regular.

**Example.**

$h(x) = x_1^2 \Rightarrow \mathcal{C} = \{x \in \mathbb{R}^2 : h(x) = 0\} = \{(x_1, x_2) : x_1 = 0\}$   
No points in  $\mathcal{C}$  are regular.

---

## Lagrange Multiplier Theorem: Sufficient Conditions

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$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ & x \in \mathbb{R}^n \end{array}$$

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i h_i(x)$$

**Theorem.** Assume that  $f(\cdot)$  and  $h(\cdot)$  are twice continuously differentiable, and that  $x^* \in \mathbb{R}^n$  and  $\lambda^* \in \mathbb{R}^m$  satisfy

$$\begin{aligned} \nabla_x L(x^*, \lambda^*) &= 0, & \nabla_\lambda L(x^*, \lambda^*) &= 0, \\ d^\top \nabla_{xx}^2 L(x^*, \lambda^*) d &> 0, & \forall d \in \mathcal{V}(x^*) \setminus \{0\}. \end{aligned}$$

Then,  $x^*$  is a strict local minimum.

## Sufficient Conditions: Proof

---

Clearly  $x^*$  is feasible. Suppose  $x^*$  is not a strict local minimum. Then, there exists  $\{x_k\} \subset \mathbb{R}^n$ ,  $h(x_k) = 0$ ,  $x_k \neq x^*$ ,  $x_k \rightarrow x^*$ , with  $f(x_k) \leq f(x^*)$ . Define

$$d_k = \frac{x_k - x^*}{\|x_k - x^*\|}, \quad \delta_k = \|x_k - x^*\|$$

Then  $\delta_k \rightarrow 0$ , and  $\{d_k\}$  must have a subsequence converging to some  $d$  with  $\|d\| = 1$ . WLOG, assume  $d_k \rightarrow d$ .

Then, by the mean value theorem,

$$h(x_k) = h(x) + \nabla h(\tilde{x}_k)^\top (\delta_k d_k) \Rightarrow \nabla h(\tilde{x}_k)^\top d_k = 0$$

Taking a limit as  $k \rightarrow \infty$ ,  $\nabla h(x^*)^\top d = 0$ .

## Sufficient Conditions: Proof

---

By the second order Taylor expansion (with remainder),

$$0 = h_i(x_k) = h_i(x^*) + \delta_k \nabla h_i(x^*)^\top d_k + \frac{1}{2} \delta_k^2 d_k^\top \nabla^2 h_i(\hat{x}_{i,k}) d_k$$

$$0 \geq f(x_k) - f(x^*) = \delta_k \nabla f(x^*)^\top d_k + \frac{1}{2} \delta_k^2 d_k^\top \nabla^2 f(\hat{x}_{0,k}) d_k$$

(Each  $\hat{x}_{i,k}$  is a point on the line segment between  $x_k$  and  $x$ .)

$$\Rightarrow 0 \geq \frac{1}{2} \delta_k^2 d_k^\top \left( \nabla^2 f(\hat{x}_{0,k}) + \sum_{i=1}^m \lambda_i \nabla^2 h_i(\hat{x}_{i,k}) \right) d_k$$

Dividing by  $\delta_k^2$  and taking a limit as  $k \rightarrow \infty$ , we have a contradiction.

## Lagrange Multiplier Theorem: Sensitivity Analysis

Consider a family of problems, parameterized by  $u \in \mathbb{R}^m$ :

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = u, \\ & x \in \mathbb{R}^n \end{array}$$

**Theorem.** Suppose there exists a local minimum-Lagrange multiplier pair  $(x^*, \lambda^*)$  satisfying the second order sufficient conditions when  $u = 0$ , with  $x^*$  regular. Then, there exists a neighborhood  $N_\epsilon(0) \subset \mathbb{R}^m$  of  $u = 0$  and a function  $x^*(\cdot)$  defined on  $N_\epsilon(0)$  such that

- (i)  $x^*(0) = x^*$ , and for each  $u \in N_\epsilon(0)$ ,  $x^*(u)$  is a strict local minimum
- (ii)  $x^*(\cdot)$  is continuously differentiable
- (iii) If  $p(u) = f(x^*(u))$ , then

$$\nabla p(0) = -\lambda^*$$

## Sensitivity Analysis: Proof

(i) & (ii): Consider, for  $u \in \mathbb{R}^m$  the following system of equations in  $(x, \lambda)$ :

$$\nabla f(x) + \nabla h(x)\lambda = 0, \quad h(x) = u.$$

At  $u = 0$ , this has the gradient

$$\begin{bmatrix} \nabla_{xx}^2 L(x^*, \lambda^*) & \nabla h(x^*) \\ \nabla h(x^*)^\top & 0 \end{bmatrix}$$

which is non-singular by the second order sufficient conditions. By the implicit function theorem, we can define  $(x^*(u), \lambda^*(u))$  satisfying the first order conditions for all  $u$  in some  $N_\epsilon(0)$ . Second order sufficient conditions follow for  $u$  sufficiently close to zero from continuity assumptions.

(iii): Note that, for  $u \in N_\epsilon(0)$ ,

$$\nabla x^*(u) \nabla f(x^*(u)) + \nabla x^*(u) \nabla h(x^*(u)) \lambda^*(u) = 0$$

Differentiating  $h(x^*(u)) = u$ ,

$$I = \nabla_u \{h(x^*(u))\} = \nabla x^*(u) \nabla h(x^*(u))$$

Then,

$$\nabla p(u) = \nabla x^*(u) \nabla f(x^*(u)) = -\lambda^*(u)$$

## Application: Portfolio Optimization

---

Consider the portfolio optimization problem without short sale constraints:

$$\begin{aligned} \sigma^2 = \text{minimize} \quad & x^\top \Gamma x \\ \text{subject to} \quad & \mathbf{1}^\top x = 1, \\ & \mu^\top x = \bar{\mu}, \\ & x \in \mathbb{R}^n \end{aligned}$$

Here, we assume that  $\Gamma \succ 0$  and  $\mathbf{1}$  and  $\mu$  are linearly independent. First order conditions:

$$2\Gamma x^* + \lambda_1^* \mathbf{1} + \lambda_2^* \mu = 0, \quad \mathbf{1}^\top x^* = 1, \quad \mu^\top x^* = \bar{\mu}$$

Then,

$$\begin{aligned} x^* &= -\frac{1}{2} \Gamma^{-1} \mathbf{1} \lambda_1^* - \frac{1}{2} \Gamma^{-1} \mu \lambda_2^* \\ 1 &= \mathbf{1}^\top x^* = -\frac{1}{2} \mathbf{1}^\top \Gamma^{-1} \mathbf{1} \lambda_1^* - \frac{1}{2} \mathbf{1}^\top \Gamma^{-1} \mu \lambda_2^* \\ \bar{\mu} &= \mu^\top x^* = -\frac{1}{2} \mu^\top \Gamma^{-1} \mathbf{1} \lambda_1^* - \frac{1}{2} \mu^\top \Gamma^{-1} \mu \lambda_2^* \end{aligned}$$

The system of equations for  $(\lambda_1^*, \lambda_2^*)$  is non-singular if  $\mathbf{1}$  and  $\mu$  are linearly independent and  $\Gamma \succ 0$ , so

$$\lambda_1^* = \eta_1 + \zeta_1 \bar{\mu}, \quad \lambda_2^* = \eta_2 + \zeta_2 \bar{\mu}$$

for some scalars  $\eta_1, \eta_2, \zeta_1, \zeta_2$  (depending on  $\Gamma$  and  $\mu$ )

$$\Rightarrow x^* = \bar{\mu}v + w$$

for some vectors  $v, w$  (depending on  $\Gamma$  and  $\mu$ )

$$\Rightarrow \sigma^2 = (\bar{\mu}v + w)^\top \Gamma (\bar{\mu}v + w) = (\alpha \bar{\mu} + \beta)^2 + \gamma$$

for some scalars  $\alpha, \beta, \gamma$  (depending on  $\Gamma$  and  $\mu$ )

B9824 Foundations of Optimization

## Lecture 3: Local Theory of Optimization II

Fall 2008

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### Outline

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1. Necessary conditions for inequality constraints
2. KKT cookbook recipe
3. Sufficient conditions for inequality constraints, sensitivity analysis
4. Geometric interpretation

Consider

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && h_1(x) = 0, \dots, h_m(x) = 0, \\ & && g_1(x) \leq 0, \dots, g_r(x) \leq 0, \\ & && x \in \mathbb{R}^n \end{aligned}$$

where

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad h_i : \mathbb{R}^n \rightarrow \mathbb{R}, \quad g_j : \mathbb{R}^n \rightarrow \mathbb{R}$$

- Assume that  $f(\cdot)$ ,  $\{h_i(\cdot)\}$ ,  $\{g_j(\cdot)\}$  are continuously differentiable on  $\mathbb{R}^n$
- The necessary and sufficient conditions are also true if these functions are just defined and continuously differentiable in a neighborhood of the local minimum

## Reduction to Equality Constraints

---

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize} \quad f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to} \quad h(x) = 0, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & \quad g(x) \leq 0, \\ & \quad x \in \mathbb{R}^n \end{array}$$

**Definition.** Given a feasible point  $x^* \in \mathbb{R}^n$ , the set of **active inequality constraints**  $\mathcal{A}(x^*)$  is defined by

$$\mathcal{A}(x^*) \triangleq \{j : g_j(x^*) = 0\} \subset \{1, \dots, r\}$$

## Reduction to Equality Constraints

---

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\ & x \in \mathbb{R}^n \end{array}$$

**Lemma.** Let  $x^*$  be a local minimum for the inequality constrained program. Then, it is also a local minimum for the equality constrained program

$$\begin{array}{ll} \text{minimize } & f(x) \\ \text{subject to } & h(x) = 0, \\ & g_j(x) = 0, \quad j \in \mathcal{A}(x^*) \\ & x \in \mathbb{R}^n \end{array}$$

## Reduction to Equality Constraints: Proof

---

Suppose  $x^*$  is not a local minimum for the equality constraint program (ECP). Then, there is a sequence of points  $\{x_k\}$  feasible for (ECP), such that  $x_k \rightarrow x^*$ , and  $f(x_k) < f(x^*)$ .

Since  $g(\cdot)$  is continuous, we have  $g(x_k) \rightarrow g(x^*)$ . In particular, if  $j \notin \mathcal{A}(x^*)$ ,

$$g_j(x_k) \rightarrow g_j(x^*) < 0$$

Thus, for  $k$  sufficiently large,  $g_j(x_k) < 0$  and  $x_k$  is feasible for the inequality constrained program (ICP). This contradicts the local optimality of  $x^*$  for (ICP).

$$\begin{array}{ll}
 f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\
 h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\
 g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\
 & x \in \mathbb{R}^n
 \end{array}$$

**Definition.** A point  $x^* \in \mathbb{R}^n$  is a **regular point** if it is feasible and if the set of constraint gradients

$$\{\nabla h_i(x^*) : 1 \leq i \leq m\} \cup \{\nabla g_j(x^*) : j \in \mathcal{A}(x^*)\}$$

are linearly independent.

**Definition.** The cone  $\mathcal{V}^{\text{EQ}}(x^*)$  at a point  $x^* \in \mathbb{R}^n$  is the set of vectors  $d \in \mathbb{R}^n$  such that

$$\nabla h_i(x^*)^\top d = 0, \quad \forall 1 \leq i \leq m, \quad \nabla g_j(x^*)^\top d = 0, \quad \forall j \in \mathcal{A}(x^*).$$

## Karush-Kuhn-Tucker Theorem: Necessary Conditions

$$\begin{array}{ll}
 f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\
 h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\
 g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\
 & x \in \mathbb{R}^n
 \end{array}$$

**Theorem.** If  $x^*$  is a local minimum that is a regular point, then there exists a unique Lagrange multiplier vectors  $\lambda^* \in \mathbb{R}^m$  and  $\mu^* \in \mathbb{R}^r$ , such that

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^r \mu_j^* \nabla g_j(x^*) = 0,$$

$$\mu_j^* \geq 0, \quad \forall 1 \leq j \leq r, \quad \mu_j^* = 0, \quad \forall j \notin \mathcal{A}(x^*).$$

If, in addition,  $f(\cdot)$ ,  $h(\cdot)$ , and  $g(\cdot)$  are twice continuously differentiable, for all  $d \in \mathcal{V}^{\text{EQ}}(x^*)$ ,

$$d^\top \left( \nabla^2 f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla^2 h_i(x^*) + \sum_{j=1}^r \mu_j^* \nabla^2 g_j(x^*) \right) d \geq 0$$

## Necessary Conditions: Proof

Everything follows by apply the necessary conditions from the Lagrange multiplier theorem to the equality constrained program defined by the active constraints, except the assertion that  $\mu_j^* \geq 0$ , for  $j \in \mathcal{A}(x^*)$ .

Suppose this does not hold for some  $j$ . Then, let  $\mathcal{C}_j \subset \mathbb{R}^n$  be the set of points feasible for all other active constraints, and  $\mathcal{V}_j^{\text{EQ}}(x^*)$  the corresponding cone of first order feasible directions. By regularity, there exists  $d \in \mathcal{V}_j^{\text{EQ}}(x^*)$  with  $\nabla g_j(x^*)^\top d < 0$ .

Then, there exists a curve  $x(t) \in \mathcal{C}_j$  with  $x(0) = x^*$ ,  $\dot{x}(0) = d$ . For small  $t$ ,  $x(t)$  is feasible and if  $\ell(t) \triangleq f(x(t))$

$$\begin{aligned}\dot{\ell}(0) &= d^\top \nabla f(x^*) = -d^\top (\nabla h(x^*)\lambda^* - \nabla g(x^*)\mu^*) \\ &= -d^\top \nabla g_j(x^*)\mu_j^* < 0,\end{aligned}$$

contradicting the local optimality of  $x^*$ .

## KKT Theorem: Interpretation

**Definition.** The **Lagrangian** function  $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \rightarrow \mathbb{R}$  is defined by

$$\begin{aligned}L(x, \lambda, \mu) &= f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x) \\ &= f(x) + \lambda^\top h(x) + \mu^\top g(x)\end{aligned}$$

The first order necessary conditions can be written as:

$$\begin{aligned}\nabla_x L(x^*, \lambda^*, \mu^*) &= 0, \quad h(x^*) = 0, \quad g(x^*) \leq 0, \\ \mu^* &\geq 0, \\ \mu_j^* g_j(x^*) &= 0, \quad \forall 1 \leq j \leq r\end{aligned}$$

The second order necessary conditions can be written as:

$$d^\top \nabla_{xx}^2 L(x^*, \lambda^*, \mu^*) d \geq 0, \quad \forall d \in \mathcal{V}^{\text{EQ}}(x^*)$$

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\ & x \in \mathbb{R}^n \end{array}$$

$$L(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

- (i) Check that a global minima exists
- (ii) Find the set of  $(x^*, \lambda^*, \mu^*)$  satisfying the necessary conditions
$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \quad h(x^*) = 0, \quad g(x^*) \leq 0,$$
$$\mu^* \geq 0, \quad \mu_j^* g_j(x^*) = 0, \quad \forall 1 \leq j \leq r$$
- (iii) Find the set of non-regular points
- (iv) The global minima must be among the points in (ii) and (iii)

(Assuming  $f(\cdot)$ ,  $h(\cdot)$ ,  $g(\cdot)$  continuously differentiable on  $\mathbb{R}^n$ )

## Example

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$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \\ \text{subject to} & x_1 + x_2 + x_3 \leq -3, \\ & x \in \mathbb{R}^3 \end{array}$$

Objective & constraints continuously differentiable

Global minima exist (coerciveness)

First order conditions:

$$\begin{array}{ll} x_1^* + \mu^* = 0 & \\ x_2^* + \mu^* = 0 & \\ x_3^* + \mu^* = 0 & \\ x_1^* + x_2^* + x_3^* \leq -3 & \\ \mu^*(x_1^* + x_2^* + x_3^* + 3) = 0 & \end{array} \quad \begin{array}{l} x^* = (-1, -1, -1) \\ \mu^* = 1 \end{array}$$

All points are regular

Global minimum:  $x^* = (-1, -1, -1)$

## KKT Theorem: Sufficient Conditions

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\ & x \in \mathbb{R}^n \end{array}$$

**Theorem.** Assume that  $f(\cdot)$ ,  $h(\cdot)$  and  $g(\cdot)$  are twice continuously differentiable, and that  $x^* \in \mathbb{R}^n$ ,  $\lambda^* \in \mathbb{R}^m$ ,  $\mu^* \in \mathbb{R}^r$  satisfy

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \quad h(x^*) = 0, \quad g(x^*) \leq 0,$$

$$\mu_j^* \geq 0, \quad \mu_j^* = 0, \quad \forall j \notin \mathcal{A}(x^*),$$

$$d^\top \nabla_{xx}^2 L(x^*, \lambda^*) d > 0, \quad \forall d \in \mathcal{V}^{\text{EQ}}(x^*) \setminus \{0\}$$

Assume also that

$$\mu_j^* > 0, \quad \forall j \in \mathcal{A}(x^*)$$

Then,  $x^*$  is a strict local minimum.

## Sufficient Conditions: Proof

Following the equality case: Suppose  $x^*$  is not a strict local minimum. Then, there exists  $\{x_k\} \subset \mathbb{R}^n$ ,  $h(x_k) = 0$ ,  $g(x_k) \leq 0$ ,  $x_k \neq x^*$ ,  $x_k \rightarrow x^*$ , with  $f(x_k) \leq f(x^*)$ . Define

$$d_k = \frac{x_k - x^*}{\|x_k - x^*\|}, \quad \delta_k = \|x_k - x^*\|$$

Then  $\delta_k \rightarrow 0$ , and  $\{d_k\}$  must have a subsequence converging to some  $d$  with  $\|d\| = 1$ . WLOG, assume  $d_k \rightarrow d$ .

As in the equality case,  $\nabla h(x^*)^\top d = 0$ . If  $j \in \mathcal{A}(x^*)$ , by the mean value theorem,

$$g_j(x_k) - g_j(x^*) \leq 0 \quad \Rightarrow \quad \nabla g_j(x^*)^\top d \leq 0$$

If  $\nabla g_j(x^*)^\top d = 0$  for all  $j \in \mathcal{A}(x^*)$ , then  $d \in \mathcal{V}^{\text{EQ}}(x^*)$ , and we proceed as before.

Suppose for some  $j \in \mathcal{A}(x^*)$ ,  $\nabla g_j(x^*)^\top d < 0$ . Then,

$$d^\top \nabla f(x^*) = -d^\top (\nabla h(x^*)\lambda^* - \nabla g(x^*)\mu^*) > 0$$

However, by the mean value theorem,

$$f(x_k) - f(x^*) \geq 0 \quad \Rightarrow \quad \nabla f(x^*)^\top d \geq 0$$

## KKT Theorem: Sensitivity Analysis

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Consider a family of problems, parameterized by  $u \in \mathbb{R}^m$  and  $v \in \mathbb{R}^r$ :

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = u, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq v, \\ & x \in \mathbb{R}^n \end{array}$$

**Theorem.** Suppose there exists a triple  $(x^*, \lambda^*, \mu^*)$  satisfying the second order sufficient conditions when  $(u, v) = (0, 0)$ , with  $x^*$  regular. Then, there exists a neighborhood  $N$  of  $(u, v) = (0, 0)$  and a function  $x^*(\cdot, \cdot)$  defined on  $N$  such that

- (i)  $x^*(0, 0) = x^*$ , and for each  $(u, v) \in N$ ,  $x^*(u, v)$  is a strict local minimum
- (ii)  $x^*(\cdot, \cdot)$  is continuously differentiable
- (iii) If  $p(u, v) = f(x^*(u, v))$ , then

$$\nabla_u p(0, 0) = -\lambda^*, \quad \nabla_v p(0, 0) = -\mu^*$$

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\ & x \in \mathbb{R}^n \end{array}$$

Assume that  $x^*$  is a feasible point and  $d \in \mathbb{R}^n$  is a direction. For small  $\alpha > 0$ ,

$$h(x^* + \alpha d) \approx h(x^*) + \nabla h(x^*)^\top (\alpha d) = \alpha \nabla h(x^*)^\top d$$

$$g(x^* + \alpha d) \approx g(x^*) + \nabla g(x^*)^\top (\alpha d) = \alpha \nabla g(x^*)^\top d$$

**Definition.** The cone of **first order feasible variations** at a point  $x^* \in \mathbb{R}^n$  is the set

$$\mathcal{V}(x^*) = \{d \in \mathbb{R}^n : \nabla h(x^*)^\top d = 0, \nabla g_j(x^*)^\top d \leq 0, \forall j \in \mathcal{A}(x^*)\}$$

## Farkas' Lemma

---

**Lemma.** (Farkas) Consider a matrix  $A \in \mathbb{R}^{m \times n}$ . Then, a vector  $z \in \mathbb{R}^m$  satisfies

$$z^\top y \leq 0 \text{ for all } y \in \mathbb{R}^m \text{ with } A^\top y \leq 0$$

if and only if

$$z = Ax \text{ for some } x \in \mathbb{R}^n \text{ with } x \geq 0$$

## Existence of Lagrange Multipliers

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$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\ & x \in \mathbb{R}^n \end{array}$$

**Theorem.** Let  $x^*$  be a local minimum. Then there exist Lagrange multipliers  $(\lambda^*, \mu^*)$  satisfying

$$\begin{aligned} \nabla f(x^*) + \nabla h(x^*)\lambda^* + \nabla g(x^*)\mu^* &= 0, \\ \mu^* &\geq 0, \quad \mu_j^* g_j(x^*) = 0, \quad \forall 1 \leq j \leq r, \end{aligned}$$

if and only if

$$\mathcal{D}(x^*) \cap \mathcal{V}(x^*) = \emptyset.$$

## Existence of Lagrange Multipliers: Proof

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Suppose there are no equality constraints. Then,

$$\mathcal{D}(x^*) \cap \mathcal{V}(x^*) = \emptyset$$

is equivalent to

$$\nabla f(x^*)^\top d \geq 0 \text{ for all } d \text{ such that } \nabla g_j(x^*)^\top d \leq 0, \quad \forall j \in \mathcal{A}(x^*)$$

By Farkas' Lemma, this is equivalent to

$$\nabla f(x^*) + \nabla g(x^*)\mu = 0$$

for some  $\mu$  with  $\mu \geq 0$  and  $\mu_j = 0$  if  $j \notin \mathcal{A}(x^*)$ .

If there are equality constraints  $h(x) = 0$ , we can add inequality constraints  $h(x) \leq 0$  and  $-h(x) \leq 0$ .

Let  $x^*$  be a local minimum.

- $\mathcal{D}(x^*) \cap \mathcal{T}(x^*) = \emptyset$  (since  $x^*$  is a local minimum)
- $\mathcal{D}(x^*) \cap \mathcal{V}(x^*) = \emptyset$  implies the existence of Lagrange multipliers
- Quasiregularity:  $\mathcal{V}(x^*) = \mathcal{T}(x^*)$

**Corollary.** Let  $x^*$  be a local minimum that is quasiregular. Then, Lagrange multipliers exist satisfying the KKT first order necessary conditions.

## Linear Constraint Qualification

---

$$\begin{aligned} f &: \mathbb{R}^n \rightarrow \mathbb{R} \\ A &\in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \end{aligned}$$

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && Ax \leq b, \\ &&& x \in \mathbb{R}^n \end{aligned}$$

**Theorem.** Suppose that  $x^*$  is a local minimum. Then,  $x^*$  is quasiregular and Lagrange multipliers exist.

**Note:** Trivially applies to linear equality constraints also. Can be extended to linear equality constraints and concave inequality constraints.

$\mathcal{V}(x^*) \subset \mathcal{T}(x^*)$  : If  $d \in \mathcal{V}(x^*) \setminus \{0\}$ , consider the sequence defined by

$$x_k \triangleq x^* + d/k$$

If  $j \notin \mathcal{A}(x^*)$ , then

$$A_j^\top x_k < b_j + A_j^\top d/k \rightarrow b_j$$

If  $j \in \mathcal{A}(x^*)$ , then

$$A_j^\top x_k = b_j + A_j^\top d/k = b_j$$

Thus, for  $k$  sufficiently large,  $x_k$  is feasible. Further,

$$\frac{x_k - x^*}{\|x_k - x^*\|} \rightarrow \frac{d}{\|d\|}$$

$\mathcal{T}(x^*) \subset \mathcal{V}(x^*)$  : If  $d \in \mathcal{V}(x^*) \setminus \{0\}$ , let  $\{x_k\}$  be the a sequence with  $x_k \rightarrow x^*$  and

$$\frac{x_k - x^*}{\|x_k - x^*\|} \rightarrow \frac{d}{\|d\|}$$

Define

$$\zeta_k \triangleq \frac{x_k - x^*}{\|x_k - x^*\|} - \frac{d}{\|d\|}, \quad d_k \triangleq d + \|d\|\zeta_k$$

Then,  $\zeta_k \rightarrow 0$  and  $d_k \rightarrow d$ , and

$$x_k - x^* = \frac{\|x_k - x^*\|}{\|d\|} d_k$$

If  $j \in \mathcal{A}(x^*)$ , then

$$0 \geq A_j^\top (x_k - x^*) = \frac{\|x_k - x^*\|}{\|d\|} A_j^\top d_k \rightarrow A_j^\top d$$

## A General Sufficiency Condition

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$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0, \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Theorem.** Let  $x^* \in \mathbb{R}^n$  be a feasible point and  $\mu^* \in \mathbb{R}^r$  be a vector such that

$$\begin{aligned} \mu^* &\geq 0, \\ \mu_j^* &= 0, \quad \forall j \notin \mathcal{A}(x^*), \\ x^* &\in \operatorname{argmin}_{x \in \Omega} L(x, \mu^*) \end{aligned}$$

Then,  $x^*$  is a global minimum.

**Note:** No differentiability or continuity assumptions made!

## A General Sufficiency Condition: Proof

---

$$\begin{aligned} f(x^*) &= f(x^*) + (\mu^*)^\top g(x^*) \\ &= \min_{x \in \Omega} f(x) + (\mu^*)^\top g(x) \\ &\leq \min_{x \in \Omega, g(x) \leq 0} f(x) + (\mu^*)^\top g(x) \\ &\leq \min_{x \in \Omega, g(x) \leq 0} f(x) \end{aligned}$$

B9824 Foundations of Optimization

## Lecture 4: Convexity

Fall 2008

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### Outline

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1. Convex sets
2. Convex functions
3. Separating/supporting hyperplanes

Consider a set  $\mathcal{C} \subset \mathbb{R}^n$ .

**Definition.** The set  $\mathcal{C}$  is **affine** if, for all points  $x_1, x_2 \in \mathcal{C}$ , and a scalar  $\lambda \in \mathbb{R}$ ,

$$\lambda x_1 + (1 - \lambda)x_2 \in \mathcal{C}$$

**Example.** The empty set is affine. Any line is affine. Any subspace is affine.

**Example.** If  $\mathcal{C}$  is the solution to a set of linear equations, e.g.

$$\mathcal{C} = \{x \in \mathbb{R}^n : Ax = b\},$$

for some matrix  $A \in \mathbb{R}^{m \times n}$  and vector  $b \in \mathbb{R}^m$ , then  $\mathcal{C}$  is an affine set.

**Definition.** Given a set of points  $\mathcal{X} \subset \mathbb{R}^n$ , the **affine hull aff**  $\mathcal{X}$  is the set of points

$$\lambda_1 x_1 + \cdots + \lambda_k x_k,$$

where  $k \geq 1$ ,  $\{x_i\} \subset \mathcal{X}$ , and

$$\lambda_1 + \cdots + \lambda_k = 1.$$

The affine hull **aff**  $\mathcal{X}$  is affine, and is the smallest affine set containing  $\mathcal{X}$ .

**Definition.** The set  $\mathcal{C}$  is **convex** if, for all points  $x_1, x_2 \in \mathcal{C}$ , and scalars  $0 \leq \lambda \leq 1$ ,

$$\lambda x_1 + (1 - \lambda)x_2 \in \mathcal{C}$$

**Note:** Clearly affine sets are also convex.

**Definition.** Given a set of points  $\mathcal{X} \subset \mathbb{R}^n$ , the **convex hull**  $\text{conv } \mathcal{X}$  is the set of points

$$\lambda_1 x_1 + \cdots + \lambda_k x_k,$$

where  $k \geq 1$ ,  $\{x_i\} \subset \mathcal{X}$ ,  $\lambda \geq 0$ , and

$$\lambda_1 + \cdots + \lambda_k = 1.$$

The convex hull  $\text{conv } \mathcal{X}$  is convex, and is the smallest convex set containing  $\mathcal{X}$ .

## Hyperplanes and Halfspaces

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**Definition.** A **hyperplane** is a set of the form

$$\{x \in \mathbb{R}^n : a^\top x = b\},$$

where  $a \in \mathbb{R}^n \setminus \{0\}$  is a non-zero vector called the **normal** vector and  $b \in \mathbb{R}$  is a scalar.

Hyperplanes are affine and thus convex.

**Definition.** A **halfspace** is a set of the form

$$\{x \in \mathbb{R}^n : a^\top x \leq b\},$$

where  $a \in \mathbb{R}^n \setminus \{0\}$  is a non-zero vector and  $b \in \mathbb{R}$  is a scalar.

Halfspaces are not affine but are convex.

**Definition.** A **norm** is a real-valued function  $\|\cdot\|_*$  on  $\mathbb{R}^n$  such that

- $\|x\|_* = 0$  if and only if  $x = 0$
- For all  $x \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ ,  $\|\lambda x\|_* = |\lambda| \|x\|_*$
- For all  $x_1, x_2 \in \mathbb{R}^n$ ,  $\|x_1 + x_2\|_* \leq \|x_1\|_* + \|x_2\|_*$

**Example.**

$$\|x\| = \|x\|_2 = \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} = \sqrt{x^\top x}$$

$$\|x\|_\Gamma = \sqrt{x^\top \Gamma x}, \quad \Gamma \text{ symmetric positive definite}$$

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1 \quad \|x\|_\infty = \max(|x_1|, \dots, |x_n|)$$

## Norm Balls

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Given a norm  $\|\cdot\|_*$ , the (closed) ball with center  $x_0 \in \mathbb{R}^n$  and radius  $r \geq 0$ ,

$$\{x \in \mathbb{R}^n : \|x - x_0\|_* \leq r\}$$

is convex.

**Example.**  $\|\cdot\|_2 \Rightarrow$  spheres are convex

**Example.**  $\|\cdot\|_\Gamma \Rightarrow$  ellipsoids are convex

**Example.**  $\|\cdot\|_\infty, \|\cdot\|_1 \Rightarrow$  'boxes' are convex

## Elementary Properties of Convex Sets

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**Theorem.** (Scalar Multiplication) If  $\mathcal{C} \subset \mathbb{R}^n$  is a convex set, and  $\alpha \in \mathbb{R}$  is a scalar, then the set

$$\alpha\mathcal{C} \triangleq \{\alpha x : x \in \mathcal{C}\}$$

is also convex.

**Theorem.** (Vector Sum) If  $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$  are convex sets, then the set

$$\mathcal{C} + \mathcal{D} \triangleq \{x + y : x \in \mathcal{C}, y \in \mathcal{D}\}$$

is also convex.

**Theorem.** (Affine Transformations) If  $\mathcal{C} \subset \mathbb{R}^n$  is a convex set,  $A \in \mathbb{R}^{m \times n}$  a matrix, and  $b \in \mathbb{R}^m$  a vector, then the set

$$\{Ax + b : x \in \mathcal{C}\}$$

is a convex subset of  $\mathbb{R}^m$ .

## Elementary Properties of Convex Sets

---

**Theorem.** (Intersection) If  $\mathcal{K}$  is an arbitrary collection of convex sets, then the intersection

$$\bigcap_{\mathcal{C} \in \mathcal{K}} \mathcal{C}$$

is also convex.

**Definition.** A set  $\mathcal{P}$  is a **polyhedron** if it is of the form

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b\},$$

for a matrix  $A \in \mathbb{R}^{m \times n}$  and a vector  $b \in \mathbb{R}^m$ .

**Note:** Linear equality constraints are also trivially allowed.

Polyhedra are convex.

**Example.** The non-negative orthant  $\{x \in \mathbb{R}^n : x \geq 0\}$ .

**Example.** The unit simplex  $\{x \in \mathbb{R}^n : x \geq 0, \mathbf{1}^\top x \leq 1\}$ .

**Example.** The probability simplex  $\{x \in \mathbb{R}^n : x \geq 0, \mathbf{1}^\top x = 1\}$ .

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## Optimization Over Convex Sets: Necessary Condition

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$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

**Theorem.** Suppose that  $\mathcal{C}$  is a convex set,  $x^*$  is a local minimum, and  $f(\cdot)$  is continuously differentiable in a neighborhood of  $x^*$ . Then, for all  $x \in \mathcal{C}$ ,

$$\nabla f(x^*)^\top (x - x^*) \geq 0$$

Suppose there exists  $x \in \mathcal{C}$  with

$$\nabla f(x^*)^\top (x - x^*) < 0$$

By the mean value theorem, for a given  $\epsilon > 0$  sufficiently small,

$$f(x^* + \epsilon(x - x^*)) = f(x^*) + \epsilon \nabla f(x^* + s\epsilon(x - x^*))^\top (x - x^*),$$

for some  $s \in [0, 1]$ . Since  $\nabla f$  is continuous, for  $\epsilon$  sufficiently small,

$$\nabla f(x^* + s\epsilon(x - x^*))^\top (x - x^*) < 0$$

Then,

$$f(x^* + \epsilon(x - x^*)) < f(x^*)$$

Since  $\mathcal{C}$  is convex, this contradicts local optimality of  $x^*$ .

## Convex Functions

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**Definition.** Let  $\mathcal{X} \subset \mathbb{R}^n$  be a convex set. A real-valued function  $f : \mathcal{X} \rightarrow \mathbb{R}$  is **convex** if, for all  $x_1, x_2 \in \mathcal{X}$  and  $\lambda \in [0, 1]$ ,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

$f(\cdot)$  is **strictly convex** if, in addition,

$$f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2)$$

when  $x_1 \neq x_2$  and  $\lambda \in (0, 1)$ .

**Note:** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex when restricted to a (convex) subset  $\mathcal{X} \subset \mathbb{R}^n$ , we will say  $f(\cdot)$  is convex over  $\mathcal{X}$ .

**Definition.** A function  $f(\cdot)$  is **concave** if  $-f(\cdot)$  is convex, it is **strictly concave** if  $-f(\cdot)$  is strictly convex.

## Extended-Value Functions

---

Given a convex function  $f(\cdot)$  with (convex) domain  $\mathcal{X} \subset \mathbb{R}^n$ , we can define the **extended-value extension**  $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{X} \\ \infty & \text{otherwise} \end{cases}$$

We define

$$\mathbf{dom} \tilde{f} \triangleq \{x \in \mathbb{R}^n : f(x) < \infty\}$$

**Definition.** An extended-value function  $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  is convex if

- the set **dom**  $g$  is convex
- for all  $x_1, x_2 \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ ,

$$g(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda g(x_1) + (1 - \lambda)g(x_2)$$

We will sometimes implicitly identify convex functions with their extended-value extensions.

## Indicator Functions

---

**Definition.** Given a set  $\mathcal{C} \subset \mathbb{R}^n$ , the **indicator function**  $I_{\mathcal{C}} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  is defined by

$$I_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ \infty & \text{otherwise} \end{cases}$$

If  $\mathcal{C}$  is a convex set, then  $I_{\mathcal{C}}(\cdot)$  is a convex function.

**Theorem.** Let  $\mathcal{C} \subset \mathbb{R}^n$  be a convex set and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  a differentiable function. Then,

(i)  $f(\cdot)$  is convex over  $\mathcal{C}$  if and only if

$$f(x_1) \geq f(x_0) + \nabla f(x_0)^\top (x_1 - x_0), \quad \forall x_0, x_1 \in \mathcal{C}$$

(ii)  $f(\cdot)$  is strictly convex over  $\mathcal{C}$  if and only if the inequality is strict when  $x_0 \neq x_1$

## First-Order Conditions: Proof

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Suppose that the inequality in (i) holds. Suppose  $x, y \in \mathcal{C}$ ,  $\lambda \in [0, 1]$ , and  $z = \lambda x + (1 - \lambda)y$ . Then,

$$f(x) \geq f(z) + \nabla f(z)^\top (x - z), \quad f(y) \geq f(z) + \nabla f(z)^\top (y - z),$$

Thus,

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(z) + \nabla f(z)^\top (\lambda x + (1 - \lambda)y - z) = f(z)$$

Conversely, suppose  $f(\cdot)$  is convex. Define, for  $x, z \in \mathcal{C}$ ,  $x \neq z$ ,  $\lambda \in (0, 1)$

$$g(\lambda) \triangleq \frac{f(x + \lambda(z - x)) - f(x)}{\lambda}$$

Note that, by convexity,  $g(\cdot)$  is monotonically increasing.

Then,

$$\nabla f(x)^\top (z - x) = \lim_{\lambda \downarrow 0} g(\lambda) \leq g(1) = f(z) - f(x)$$

(ii) is proved the same way.

## Second-Order Conditions

---

**Theorem.** Let  $\mathcal{C} \subset \mathbb{R}^n$  be a convex set and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  a twice continuously differentiable function. Then,

- (i) if  $\nabla^2 f(x) \succeq 0$  for all  $x \in \mathcal{C}$ ,  $f(\cdot)$  is convex over  $\mathcal{C}$
- (ii) if  $\nabla^2 f(x) \succ 0$  for all  $x \in \mathcal{C}$ ,  $f(\cdot)$  is strictly convex over  $\mathcal{C}$

Note: Previous theorems can be applied if  $f : \mathcal{C} \rightarrow \mathbb{R}$ ,  $\mathcal{C}$  is convex, and differentiability assumptions hold on an open set containing  $\mathcal{C}$  (e.g., if  $\mathcal{C}$  itself is open).

(i) By the second order Taylor expansion, if  $x, y \in \mathcal{C}$ ,

$f(y) = f(x) + \nabla f(x)^\top (y - x) + \frac{1}{2}(y - x)^\top \nabla^2 f(x + \epsilon(y - x))(y - x)$ ,  
for some  $\epsilon \in [0, 1]$ . If  $\nabla^2 f \succeq 0$ ,

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

Convexity follows from the first order conditions.

(ii) is proved the same way.

## Examples

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- *Exponential.*  $e^{ax}$  is convex on  $\mathbb{R}$ , for any  $a \in \mathbb{R}$ .
- *Powers.*  $x^a$  is convex on  $(0, \infty)$ , for any  $a \geq 1$  or  $a \leq 0$ , and concave for  $0 \leq a \leq 1$ .
- *Powers of absolute value.*  $|x|^p$  is convex on  $\mathbb{R}$ , for any  $p \geq 1$ .
- *Logarithm.*  $\log x$  is concave on  $(0, \infty)$
- *Negative entropy.*  $x \log x$  is convex on  $(0, \infty)$ , or convex on  $[0, \infty)$  if we set  $0 \log 0 = 0$ .

## Examples

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- *Affine functions.* Any affine function

$$f(x) \triangleq a^\top x + b$$

is concave and convex on  $\mathbb{R}^n$ .

- *Norms.* Any norm  $\|\cdot\|_*$  on  $\mathbb{R}^n$  is convex.

- *Log-sum-exp.* The function

$$f(x) \triangleq \log(e^{x_1} + \dots + e^{x_n})$$

is convex on  $\mathbb{R}^n$ .

- *Geometric mean.* The function

$$f(x) \triangleq \left( \prod_{i=1}^n x_i \right)^{\frac{1}{n}}$$

is concave on  $(0, \infty)^n$ .

## Operations That Preserve Convexity

---

- *Nonnegative multiples.* If  $f(\cdot)$  is convex and  $w \geq 0$ , then

$$g(x) \triangleq wf(x)$$

is convex.

- *Sums.* If  $f_1(\cdot)$  and  $f_2(\cdot)$  are convex, then

$$g(x) \triangleq f_1(x) + f_2(x)$$

is convex.

- *Nonnegative weighted sums.* If  $f_1(\cdot), \dots, f_k(\cdot)$  are convex and  $w_i \geq 0$ , then

$$g(x) \triangleq w_1 f_1(x) + \dots + w_k f_k(x)$$

is convex.

## Operations That Preserve Convexity

---

- *Pointwise maxima.* If  $f_i(\cdot)$  is convex, for  $i \in \mathcal{I}$ , then

$$g(x) \triangleq \sup_{i \in \mathcal{I}} f_i(x)$$

is an extended-value convex function.

- *Minimization.* If  $f(x, y)$  is convex over  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ , and  $\mathcal{C} \subset \mathbb{R}^m$  is convex, then

$$g(x) \triangleq \inf_{y \in \mathcal{C}} f(x, y)$$

is convex provided  $g(x) > -\infty$ .

- *Composition with affine functions.* If  $f(\cdot)$  is convex, then

$$g(x) \triangleq f(Ax + b)$$

is convex.

## Strategies to Verify Convexity

---

- Construct from known convex functions using operations that preserve convexity
- Use first- or second-order differentiability properties of convex functions
- Restrict to a line, e.g.  $f(\cdot)$  is convex over  $\mathcal{C}$  if and only if, for every  $x_1, x_2 \in \mathcal{C}$ ,

$$g(t) \triangleq f(x_1 + t(x_2 - x_1))$$

is convex over  $[0, 1]$

- Directly verify using the definition of convexity

If  $f(\cdot)$  is convex over a convex set  $\mathcal{C} \subset \mathbb{R}^n$ , every sublevel set

$$\{x \in \mathcal{C} : f(x) \leq \gamma\}$$

is a convex subset of  $\mathbb{R}^n$ .

**Note:** The converse is not true! For example,  $\log x$  is not convex on  $(0, \infty)$ , however every sublevel set is convex.

## Optimality for Convex Optimization

---

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

**Theorem.** Suppose that  $\mathcal{C} \subset \mathbb{R}^n$  is convex,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex over  $\mathcal{C}$ .

- (i) any local minimum of  $f(\cdot)$  is also a global minimum
- (ii) if  $f(\cdot)$  is strictly convex, then there exists at most one global minimum

(i) Suppose  $x^*$  is a local minimum, and there exists some  $x \neq x^*$  with  $f(x) < f(x^*)$ . Then, if  $\lambda \in [0, 1)$ ,

$$f(\lambda x^* + (1 - \lambda)x) \leq \lambda f(x^*) + (1 - \lambda)f(x) < f(x^*)$$

This contradicts local optimality.

(ii) Suppose  $x_0 \neq x_1$  are two global minima. Then,

$$f\left(\frac{1}{2}(x_0 + x_1)\right) < \frac{f(x_0) + f(x_1)}{2}$$

This contradicts the global optimality of  $x_0$  and  $x_1$ .

## Necessary & Sufficient Optimality Condition

---

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \subset \mathbb{R}^n \end{array}$$

**Theorem.** Suppose that  $\mathcal{C} \subset \mathbb{R}^n$  is convex,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex over  $\mathcal{C}$  and differentiable, and  $x^* \in \mathcal{C}$  is a feasible point. Then,  $x^*$  is a global minimum if and only if

$$\nabla f(x^*)^\top (x - x^*) \geq 0, \quad \forall x \in \mathcal{C}$$

**Proof.** Necessity follows from earlier theorem, since  $\mathcal{C}$  is convex. For sufficiency, note that

$$f(x) \geq f(x^*) + \nabla f(x^*)^\top (x - x^*) \geq f(x^*), \quad \forall x \in \mathcal{C}$$

□

# Projection Theorem

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Let  $\mathcal{C} \subset \mathbb{R}^n$  be a **closed** and non-empty convex set, and  $\|\cdot\| = \|\cdot\|_2$  the Euclidean norm. Fix the vector  $x \in \mathbb{R}^n$ .

$$\begin{aligned} & \text{minimize} && \|z - x\| \\ & \text{subject to} && z \in \mathcal{C} \subset \mathbb{R}^n \end{aligned}$$

**Theorem.** For every  $x \in \mathbb{R}^n$ , the optimization problem has a unique global minimum  $x^*$  called the **projection** of  $x$  onto  $\mathcal{C}$ . A vector  $x' \in \mathcal{C}$  is equal to  $x^*$  if and only if

$$(x - x')^\top (z - x') \leq 0, \quad \forall z \in \mathcal{C}$$

## Projection Theorem: Proof

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Existence follows from the fact that  $\|z - x\|$  is coercive,  $\mathcal{C}$  is closed.

Uniqueness follows since we can equivalently minimize  $f(z) \triangleq \|z - x\|^2$ , and

$$f(x) = (z - x)^\top (z - x) = z^\top z - 2z^\top x + x^\top x$$

is strictly convex.

Necessary and sufficient conditions follow from the fact that

$$\nabla f(x^*) = 2(x^* - x)$$

**Definition.** The hyperplane  $\{x \in \mathbb{R}^n : a^\top x = b\}$  **supports** the convex set  $\mathcal{C} \subset \mathbb{R}^n$  at the point  $\bar{x}$  if  $a^\top \bar{x} = b$  and

$$a^\top x \geq b, \quad \forall x \in \mathcal{C}$$

**Theorem.** Let  $\mathcal{C} \subset \mathbb{R}^n$  be a convex set and  $\bar{x} \in \mathbb{R}^n$  be a point that is not in the interior of  $\mathcal{C}$ . Then, there exists a supporting hyperplane at  $\bar{x}$ , that is, a vector  $a \in \mathbb{R}^n$ ,  $a \neq 0$ , such that

$$a^\top x \geq a^\top \bar{x}, \quad \forall x \in \mathcal{C}$$

## Supporting Hyperplane Theorem: Proof

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Set  $\bar{\mathcal{C}} = \text{cl } \mathcal{C}$ , note that  $\bar{\mathcal{C}}$  is closed, convex, and non-empty.

Let  $\{x_k\}$  be a sequence of points such that  $x_k \notin \bar{\mathcal{C}}$ , and  $x_k \rightarrow \bar{x}$ . This sequence exists since  $\bar{x}$  is not an interior point.

For each  $x_k$ , let  $\hat{x}_k$  be the projection of  $x_k$  onto  $\bar{\mathcal{C}}$ . Then,

$$(\hat{x}_k - x_k)^\top (x - \hat{x}_k) \geq 0, \quad \forall x \in \bar{\mathcal{C}}$$

Then, for all  $k$  and  $x \in \bar{\mathcal{C}}$ ,

$$\begin{aligned} (\hat{x}_k - x_k)^\top x &\geq (\hat{x}_k - x_k)^\top \hat{x}_k = (\hat{x}_k - x_k)^\top (\hat{x}_k - x_k) + (\hat{x}_k - x_k)^\top x_k \\ &\geq (\hat{x}_k - x_k)^\top x_k \end{aligned}$$

Set

$$a_k \triangleq \frac{\hat{x}_k - x_k}{\|\hat{x}_k - x_k\|},$$

then

$$a_k^\top x \geq a_k^\top x_k, \quad \forall k, x \in \bar{C}$$

Since  $\|a_k\| = 1$ , the sequence  $\{a_k\}$  has a non-zero subsequential limit  $a$ , and

$$a^\top x \geq a^\top \bar{x}, \quad \forall k, x \in \bar{C}$$

## Separating Hyperplane Theorem

---

**Theorem.** Let  $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n$  be two disjoint non-empty convex sets. There exists a hyperplane that separates them, that is a vector  $a \in \mathbb{R}^n$ ,  $a \neq 0$ , with

$$a^\top x_1 \leq a^\top x_2, \quad \forall x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2$$

**Proof.** Consider the convex set

$$\mathcal{D} = \mathcal{C}_1 - \mathcal{C}_2$$

Since the sets are disjoint,  $0 \notin \mathcal{D}$ , thus there exists a vector  $a \neq 0$  with

$$0 \leq a^\top (x_1 - x_2), \quad \forall x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2$$

□

**Theorem.** Let  $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n$  be two disjoint non-empty convex sets, with  $\mathcal{C}_1$  closed and  $\mathcal{C}_2$  compact. There exists a hyperplane that strictly separates  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , that is a vector  $a \in \mathbb{R}^n$ ,  $a \neq 0$ , and a scale  $b \in \mathbb{R}$ , with

$$a^\top x_1 < b < a^\top x_2, \quad \forall x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2$$

## Strictly Separating Hyperplanes: Proof Sketch

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Since  $\mathcal{C}_1$  is closed and  $\mathcal{C}_2$  is compact,  $\mathcal{C}_1 - \mathcal{C}_2$  is closed. Since  $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ ,  $0 \notin \mathcal{C}_1 - \mathcal{C}_2$ . A strictly separating hyperplane can be constructed as in the previous theorem.

B9824 Foundations of Optimization

## Lecture 5: Duality I

Fall 2008

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### Outline

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1. Separating/supporting hyperplanes
2. Farkas' Lemma
3. Geometric multipliers
4. Weak duality

**Definition.** The hyperplane  $\{x \in \mathbb{R}^n : \mu^\top x = b\}$  with normal vector  $\mu \in \mathbb{R}^n \setminus \{0\}$  and  $b \in \mathbb{R}$  **supports** the convex set  $\mathcal{C} \subset \mathbb{R}^n$  at the point  $\bar{x}$  if

$$\mu^\top x \geq \mu^\top \bar{x} = b, \quad \forall x \in \mathcal{C}$$

Equivalently,

$$\inf_{x \in \mathcal{C}} \mu^\top x \geq \mu^\top \bar{x} = b$$

**Theorem.** Let  $\mathcal{C} \subset \mathbb{R}^n$  be a convex set and  $\bar{x} \in \mathbb{R}^n$  be a point that is not in the interior of  $\mathcal{C}$ . Then, there exists a supporting hyperplane at  $\bar{x}$ , that is, a vector  $\mu \in \mathbb{R}^n$ ,  $\mu \neq 0$ , such that

$$\mu^\top x \geq \mu^\top \bar{x}, \quad \forall x \in \mathcal{C}$$

## Supporting Hyperplane Theorem: Proof

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Define the  $\bar{C} = \mathbf{cl} C$ , and note that  $\bar{C}$  is convex.

Let  $\{x_k\}$  be a sequence of points such that  $x_k \notin \bar{C}$ ,  $x_k \neq x$ , and  $x_k \rightarrow \bar{x}$ . This sequence exists since  $\bar{x}$  is not an interior point of  $\mathcal{C}$ .

For each  $x_k$ , let  $\hat{x}_k$  be the projection of  $x_k$  onto  $\bar{C}$  (note that  $\bar{C}$  is closed). Then,

$$(\hat{x}_k - x_k)^\top (x - \hat{x}_k) \geq 0, \quad \forall x \in \bar{C}$$

Then, for all  $k$  and  $x \in \bar{C}$ ,

$$\begin{aligned} (\hat{x}_k - x_k)^\top x &\geq (\hat{x}_k - x_k)^\top \hat{x}_k = (\hat{x}_k - x_k)^\top (\hat{x}_k - x_k) + (\hat{x}_k - x_k)^\top x_k \\ &\geq (\hat{x}_k - x_k)^\top x_k \end{aligned}$$

Set

$$\mu_k \triangleq \frac{\hat{x}_k - x_k}{\|\hat{x}_k - x_k\|},$$

then

$$\mu_k^\top x \geq \mu_k^\top x_k, \quad \forall k, x \in \bar{\mathcal{C}}$$

Since  $\|\mu_k\| = 1$ , the sequence  $\{\mu_k\}$  has a non-zero subsequential limit  $\mu$ , and

$$\mu^\top x \geq \mu^\top \bar{x}, \quad \forall k, x \in \bar{\mathcal{C}}$$

## Separating Hyperplane Theorem

---

**Theorem.** Let  $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^n$  be two disjoint non-empty convex sets. There exists a hyperplane that separates them, that is a vector  $\mu \in \mathbb{R}^n$ ,  $\mu \neq 0$ , and a scalar  $b \in \mathbb{R}$  with

$$\mu^\top x_1 \leq b \leq \mu^\top x_2, \quad \forall x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2$$

**Proof.** Consider the convex set

$$\mathcal{D} = \mathcal{C}_1 - \mathcal{C}_2 = \{x_1 - x_2 : x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2\}$$

Since the sets are disjoint,  $0 \notin \mathcal{D}$ , thus there exists a vector  $\mu \neq 0$  with

$$0 \leq \mu^\top (x_1 - x_2), \quad \forall x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2$$

□

**Theorem.** Let  $\mathcal{C} \subset \mathbb{R}^n$  be a closed convex set and  $\bar{x} \notin \mathcal{C}$  a point. Then, there exists a hyperplane that **strictly** separates  $\bar{x}$  and  $\mathcal{C}$ . In other words, there exists a vector  $\mu \in \mathbb{R}^n \setminus \{0\}$  and a scalar  $b \in \mathbb{R}$  such that

$$\mu^\top \bar{x} < b < \inf_{x \in \mathcal{C}} \mu^\top x$$

**Proof.** Define

$$r \triangleq \min_{x \in \mathcal{C}} \|x - \bar{x}\|.$$

Since  $\mathcal{C}$  is closed and  $\bar{x} \notin \mathcal{C}$ ,  $r > 0$ . Then, define

$$\bar{\mathcal{C}} = \{x \in \mathbb{R}^n : \|x - \bar{x}\| \leq r/2\}$$

Clearly  $\mathcal{C}$  and  $\bar{\mathcal{C}}$  are disjoint, so we can apply the separating hyperplane theorem. □

## Halfspace Characterization

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**Corollary.** If  $\mathcal{C} \subset \mathbb{R}^n$  is a closed convex set, then  $\mathcal{C}$  is the intersection of all the closed halfspaces that contain it.

**Proof.** Let  $\mathcal{H}$  be the collect of all of all closed halfspaces containing  $\mathcal{C}$ . Clearly

$$\mathcal{C} \subset \bigcap_{H \in \mathcal{H}} H$$

Suppose there is some  $x \in \bigcap_{H \in \mathcal{H}} H$  with  $x \notin \mathcal{C}$ . Then, there exists a hyperplane that strictly separates  $x$  and  $\mathcal{C}$ , and  $x$  does not lie in the closed halfspace containing  $\mathcal{C}$ . By contradiction,

$$\mathcal{C} \subset \bigcap_{H \in \mathcal{H}} H$$

□

**Lemma.** (Farkas) Consider a matrix  $A \in \mathbb{R}^{m \times n}$ . Given a vector  $z \in \mathbb{R}^m$ , the following conditions are equivalent:

- (i)  $z^\top y \leq 0$  for all  $y \in \mathbb{R}^m$  with  $A^\top y \leq 0$
- (ii)  $z = Ax$  for some  $x \in \mathbb{R}^n$  with  $x \geq 0$

## Farkas' Lemma: Proof

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(ii)  $\Rightarrow$  (i):  $z^\top y = xA^\top y \leq 0$ , since  $A^\top y \leq 0$  and  $x \geq 0$

(i)  $\Rightarrow$  (ii): Suppose  $z$  satisfies (i), and there is no  $x \geq 0$  with  $z = Ax$ . Define  $\mathcal{C} \triangleq \{Ax \in \mathbb{R}^m : x \geq 0\}$ .  $\mathcal{C}$  is a closed convex set, and  $z \notin \mathcal{C}$ . By the strictly separating hyperplane theorem, there exists  $y \in \mathbb{R}^m$ ,  $y \neq 0$ , with

$$y^\top z < y^\top z', \quad \forall z' \in \mathcal{C}$$

Since  $0 \in \mathcal{C}$ ,

$$y^\top z < 0$$

Now, if  $A_i$  is a column of  $A$  and  $\lambda > 0$ ,  $\lambda A_i \in \mathcal{C}$ . Thus,

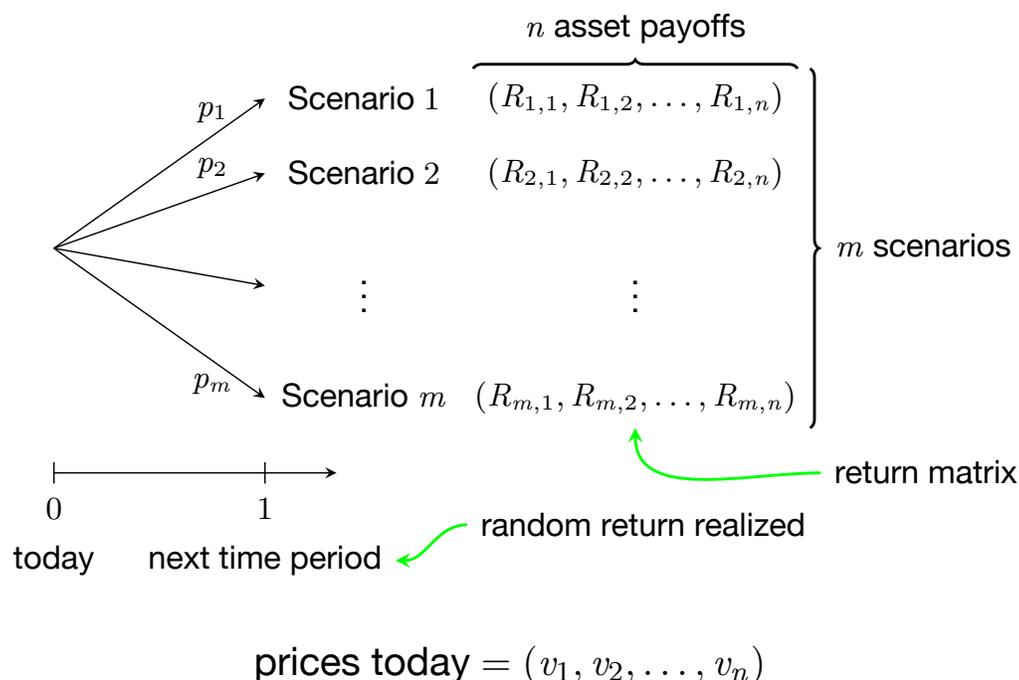
$$y^\top z < \lambda y^\top A_i$$

Dividing by  $\lambda$  and taking  $\lambda \rightarrow \infty$ ,

$$0 \leq y^\top A_i$$

Then,  $A^\top y \geq 0$ . Contradiction.

## Application: Arbitrage



## Application: Arbitrage

A **portfolio** is described by a vector  $x \in \mathbb{R}^n$ , specifying a quantity  $x_i$  of each  $i$ th asset.

$$\text{price today} = v^\top x, \quad \text{future payoffs} = Rx$$

**Definition.** An **arbitrage opportunity** is a portfolio  $x$  such that

$$v^\top x < 0, \quad Rx \geq 0$$

A **consistent** market has no arbitrage opportunities.

How can we determine if a market is consistent?

A market is consistent if and only if there exists a vector  $q \geq 0$ , with  $v = R^\top q$ , then the market is consistent.

## The Primal Problem

---

Consider the **primal** optimization problem

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

Define  $f^*$  to be the value

$$f^* = \inf_{x \in \Omega, g(x) \leq 0} f(x)$$

**Assumption.** Assume that the feasible set is non-empty and that the optimal cost is bounded below. In other words,

$$-\infty < f^* < \infty$$

**Note:** We are not making any other assumptions about  $f(\cdot)$ ,  $g(\cdot)$ , or  $\Omega$  for the moment!

## Geometric Multipliers

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Definition.** For  $x \in \Omega$  and  $\mu \in \mathbb{R}^r$ , define the **Lagrangian** function

$$L(x, \mu) \triangleq f(x) + \mu^\top g(x) = f(x) + \sum_{j=1}^r \mu_j g_j(x)$$

A vector  $\mu^* \in \mathbb{R}^r$  is a **geometric multiplier** if

- (i)  $\mu^* \geq 0$
- (ii)  $f^* = \inf_{x \in \Omega} L(x, \mu^*)$

**Definition.** The set  $\mathcal{S} \subset \mathbb{R}^{r+1}$  of **constraint-cost pairs** is defined by

$$\mathcal{S} \triangleq \{(g(x), f(x)) \in \mathbb{R}^{r+1} : x \in \Omega\}$$

**Definition.** Given a normal  $(\mu, \mu_0) \in \mathbb{R}^{r+1} \setminus \{0\}$ , define the **hyperplane** passing through  $(\bar{z}, \bar{w}) \in \mathbb{R}^{r+1}$  by

$$\{(z, w) \in \mathbb{R}^{r+1} : \mu^\top z + \mu_0 w = \mu^\top \bar{z} + \mu_0 \bar{w}\}$$

Define the **positive halfspace**

$$\{(z, w) \in \mathbb{R}^{r+1} : \mu^\top z + \mu_0 w \geq \mu^\top \bar{z} + \mu_0 \bar{w}\}$$

and the **negative halfspace**

$$\{(z, w) \in \mathbb{R}^{r+1} : \mu^\top z + \mu_0 w \leq \mu^\top \bar{z} + \mu_0 \bar{w}\}$$

The hyperplane is **non-vertical** if  $\mu_0 \neq 0$ .

**Note:** Any non-vertical hyperplane with normal  $(\mu, \mu_0)$  can be normalized so that  $\mu_0 = 1$ .

## Visualization Lemma

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**Lemma.**

- (i) The hyperplane with normal  $(\mu, 1)$  that passes through the vector  $(g(x), f(x))$  intercepts the vertical axis

$$\{(0, w) \in \mathbb{R}^{r+1} : x \in \mathbb{R}\}$$

at the level  $L(x, \mu)$

- (ii) Among all hyperplanes with normal  $(\mu, 1)$  that contain  $\mathcal{S}$  in the positive halfspace, the highest interception of the vertical axis is attained at

$$\inf_{x \in \Omega} L(x, \mu)$$

- (iii)  $\mu^*$  is a geometric multiplier if and only if  $\mu^* \geq 0$  and, among all hyperplanes with normal  $(\mu^*, 1)$  that contain  $\mathcal{S}$  in the positive halfspace, the highest interception of the vertical axis is attained at  $f^*$

(i): The hyperplane is the set of  $(z, w)$  satisfying

$$\mu^\top z + w = \mu^\top g(x) + f(x)$$

If  $z = 0$ , then we must have  $w = L(x, \mu)$ .

(ii): The hyperplane with normal  $(\mu, 1)$  that intercepts the axis at level  $c$  is the set of  $(z, w)$  with

$$\mu^\top z + w = c$$

If  $S$  lies in the positive halfspace, then

$$L(x, \mu) = f(x) + \mu^\top g(x) \geq c$$

Thus, the maximum intercept is  $c^* = \inf_{x \in \Omega} L(x, \mu)$ .

(iii): Follows from (ii) and the definition of a geometric multiplier.

## Geometric Multipliers and Optimality

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$$\begin{aligned} f &: \Omega \rightarrow \mathbb{R} \\ g &: \Omega \rightarrow \mathbb{R}^r \\ \Omega &\subset \mathbb{R}^n \end{aligned}$$

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g(x) \leq 0 \\ &&& x \in \Omega \end{aligned}$$

**Theorem.** Let  $\mu^*$  be a geometric multiplier. Then,  $x^*$  is a global minimum if and only if  $x^*$  is feasible and

$$x^* \in \operatorname{argmin}_{x \in \Omega} L(x, \mu^*), \quad \mu_j^* g_j(x^*) = 0, \quad \forall 1 \leq j \leq r$$

Assume  $x^*$  is a global minimum. Then,

$$f^* = f(x^*) \geq f(x^*) + (\mu^*)^\top g(x^*) = L(x^*, \mu^*) \geq \inf_{x \in \Omega} L(x, \mu^*) = f^*$$

Then,  $(\mu^*)^\top g(x^*) = 0$ .

Conversely,

$$f(x^*) = f(x^*) + (\mu^*)^\top g(x^*) = L(x^*, \mu^*) = \min_{x \in \Omega} L(x, \mu^*) = f^*$$

## The Dual Function

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$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Definition.** The **dual function**  $q : \mathbb{R}^r \rightarrow \mathbb{R} \cup \{-\infty\}$  is defined by

$$q(\mu) \triangleq \inf_{x \in \Omega} L(x, \mu)$$

**Note:**  $q(\mu) < \infty$  since  $\Omega$  is non-empty, by assumption. However,  $q(\mu)$  may be  $-\infty$  for some  $\mu$ . We define the domain

$$\text{dom } q = \{\mu \in \mathbb{R}^{r+1} : q(\mu) > -\infty\}$$

## The Dual Problem

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Definition.** The **dual problem** is defined by

$$\begin{array}{ll} \text{maximize } & q(\mu) \\ \text{subject to } & \mu \geq 0 \end{array}$$

The **dual optimal value** is given by

$$q^* \triangleq \sup_{\mu \geq 0} q(\mu)$$

The dual problem corresponds to finding the maximum point of interception of the vertical axis, over all hyperplanes with normal  $(\mu, 1)$ , where  $\mu \geq 0$ .

**Note:** It is possible that  $q(\mu) = -\infty$  for all  $\mu \geq 0$ . In this case, we say that the dual problem is infeasible, and set  $q^* = -\infty$ .

## The Dual Function

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

$$q(\mu) \triangleq \inf_{x \in \Omega} L(x, \mu)$$

**Theorem.** The domain **dom**  $q$  is convex, and  $q(\cdot)$  is concave over its domain.

**Proof.** Follows since  $q(\cdot)$  is a pointwise minimum of concave (in fact, linear) functions. □

**Theorem.** (Weak Duality)  $q^* \leq f^*$

**Proof.** If  $\mu \geq 0$ ,  $x \in \Omega$ , and  $g(x) \leq 0$ ,

$$q(\mu) = \inf_{z \in \Omega} L(z, \mu) \leq f(x) + \mu^\top g(x) \leq f(x)$$

Thus,

$$q^* = \sup_{\mu \geq 0} q(\mu) \leq \inf_{x \in \Omega, g(x) \leq 0} f(x) = f^*$$

□

## Duality Gap

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**Definition.** If  $q^* = f^*$  we say there is **no duality gap**. If  $q^* < f^*$ , there is a duality gap.

**Theorem.** If there is no duality gap, the set of geometric multipliers is equal to the set of dual optimal solutions.

If there is a duality gap, the set of geometric multipliers is empty.

**Proof.** By definition,  $\mu^* \geq 0$  is a geometric multiplier iff  $f^* = q(\mu^*) \leq q^*$ . By the weak duality theorem, this holds iff there is no duality gap. □

## Examples

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$$\begin{aligned} &\text{minimize} && f(x) = x_1 - x_2 \\ &\text{subject to} && g(x) = x_1 + x_2 - 1 \leq 0 \\ &&& x \in \Omega = \{(x_1, x_2) : x_1, x_2 \geq 0\} \end{aligned}$$

$$f^* = -1, \quad x^* = (0, 1)$$

$$q(\mu) = \min_{x \geq 0} x_1 - x_2 + \mu(x_1 + x_2 - 1) = \begin{cases} -\mu & \text{if } \mu \geq 1 \\ -\infty & \text{if } \mu < 1 \end{cases}$$

$$q^* = -1, \quad \mu^* = 1$$

## Examples

---

$$\begin{aligned} &\text{minimize} && f(x) = |x_1| + x_2 \\ &\text{subject to} && g(x) = x_1 \leq 0 \\ &&& x \in \Omega = \{(x_1, x_2) : x_2 \geq 0\} \end{aligned}$$

$$f^* = 0, \quad x^* = (0, 0)$$

$$q(\mu) = \min_{x_2 \geq 0} |x_1| + x_2 + \mu x_2 = \begin{cases} 0 & \text{if } |\mu| \leq 1 \\ -\infty & \text{if } |\mu| > 1 \end{cases}$$

$$q^* = 0, \quad \mu^* \in [0, 1]$$

## Examples

---

$$\begin{array}{ll} \text{minimize} & f(x) = x \\ \text{subject to} & g(x) = x^2 \leq 0 \\ & x \in \Omega = \mathbb{R} \end{array}$$

$$f^* = 0, \quad x^* = 0$$

$$q(\mu) = \min_{x \in \mathbb{R}} x + \mu x^2 = \begin{cases} -\frac{1}{4\mu} & \text{if } \mu > 0 \\ -\infty & \text{if } \mu \leq 0 \end{cases}$$

$$q^* = 0, \quad \text{no duality gap, but no optimal dual solution}$$

## Examples

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$$\begin{array}{ll} \text{minimize} & f(x) = -x \\ \text{subject to} & g(x) = x - 1/2 \leq 0 \\ & x \in \Omega = \{0, 1\} \end{array}$$

$$f^* = 0, \quad x^* = 0$$

$$q(\mu) = \min_{x \in \{0, 1\}} -x + \mu(x - 1/2) = \min(-\mu/2, \mu/2 - 1)$$

$$q^* = -1/2, \quad \mu^* = -1$$

**Theorem.**  $(x^*, \mu^*)$  is an optimal solution/geometric multiplier pair if and only if

- (i)  $x^* \in \Omega, g(x^*) \leq 0$  (primal feasibility)
- (ii)  $\mu^* \geq 0$  (dual feasibility)
- (iii)  $x^* \in \operatorname{argmin}_{x \in \Omega} L(x, \mu^*)$  (Lagrangian optimality)
- (iv)  $\mu_j^* g_j^*(x^*) = 0, \forall 1 \leq j \leq r$  (complementary slackness)

**Note:** This is only useful if there is no duality gap!

## Primal and Dual Optimality Conditions: Proof

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If  $(x^*, \mu^*)$  is an optimal solution/geometric multiplier pair, then clearly (i) and (ii) hold. (iii) and (iv) follow from the earlier theorem.

Conversely, if (i)–(iv) hold,

$$f^* \leq f(x^*) = L(x^*, \mu^*) = \min_{x \in \Omega} L(x, \mu^*) = q(\mu^*) \leq q^*$$

By weak duality, equality must hold and hence  $x^*$  is primal optimal and  $\mu^*$  is dual optimal.

**Theorem.**  $(x^*, \mu^*)$  is an optimal solution/geometric multiplier pair if and only if  $x \in \Omega$ ,  $\mu^* \geq 0$ , and  $(x^*, \mu^*)$  is a **saddle point** of the Lagrangian, in the sense that

$$L(x^*, \mu) \leq L(x^*, \mu^*) \leq L(x, \mu^*), \quad \forall x \in \Omega, \mu \geq 0$$

**Proof.** □

**Note:** This is only useful if there is no duality gap!

## Saddle Point Theorem: Proof

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If  $(x^*, \mu^*)$  is an optimal solution/geometric multiplier pair, from optimality condition (iii),

$$L(x^*, \mu^*) \leq L(x, \mu^*), \quad \forall x \in \Omega$$

Further, if  $\mu \geq 0$ , using optimality conditions (iii) and (iv),

$$L(x^*, \mu) \leq f(x^*) = L(x^*, \mu^*)$$

Conversely, assume the  $x^* \in \Omega$ ,  $\mu^* \geq 0$ , and  $(x^*, \mu^*)$  is a saddle point. Then,

$$\sup_{\mu \geq 0} L(x^*, \mu) = \sup_{\mu \geq 0} f(x^*) + \mu^\top g(x^*) = \begin{cases} f(x^*) & \text{if } g(x^*) \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Thus,  $g(x^*) \leq 0$ ,  $L(x^*, \mu^*) = f(x^*)$ , and  $\mu_j^* g_j(x^*) = 0$ ,  $\forall j$ . Thus, optimality conditions (i)–(iv) hold.

Suppose the primal problem is unbounded, that is,  $f^* = -\infty$ . Then, the proof of the weak duality theorem applies, and

$$q(\mu) = -\infty, \quad \forall \mu \geq 0$$

Thus, the dual problem is infeasible.

Alternatively, assume that the primal problem is infeasible. In general, nothing can be said about the dual problem.

For linear programs, primal infeasibility implies dual unboundedness.

## Equality Constraints

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The theory developed thus far can be extended to equality constraints by introducing a pair of inequality constraints for each equality constraint. Equivalently, we can eliminate non-negativity constraints on the multipliers for equality constraints.

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \Omega \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0 \\ g : \Omega \rightarrow \mathbb{R}^r & g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Definition.** For  $x \in \Omega$ ,  $\lambda \in \mathbb{R}^m$ ,  $\mu \in \mathbb{R}^r$ , define the **Lagrangian** function

$$L(x, \mu) \triangleq f(x) + \lambda^\top h(x) + \mu^\top g(x) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^r \mu_j g_j(x)$$

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \Omega \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0 \\ g : \Omega \rightarrow \mathbb{R}^r & g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Definition.**  $(\lambda^*, \mu^*) \in \mathbb{R}^m \times \mathbb{R}^r$  is a **geometric multiplier** if

- (i)  $\mu^* \geq 0$
- (ii)  $f^* = \inf_{x \in \Omega} L(x, \lambda^*, \mu^*)$

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \Omega \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0 \\ g : \Omega \rightarrow \mathbb{R}^r & g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Definition.** The **dual function**  $q : \mathbb{R}^m \times \mathbb{R}^r \rightarrow \mathbb{R} \cup \{-\infty\}$  is defined by

$$q(\lambda, \mu) \triangleq \inf_{x \in \Omega} L(x, \lambda, \mu)$$

The **dual problem** is defined by

$$\begin{array}{ll} \text{maximize} & q(\lambda, \mu) \\ \text{subject to} & \mu \geq 0 \\ & \lambda \in \mathbb{R}^m \end{array}$$

This theory is most useful when there is no duality gap, or, when geometric multipliers exist.

This will require:

- Convexity of the objective and constraints
- Technical conditions, similar to constraint qualification

B9824 Foundations of Optimization

## Lecture 6: Duality II

Fall 2008

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### Outline

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1. Duality and decentralization
2. Duality and combinatorial optimization
3. Strong duality

Suppose that the decision variables  $x$  decompose according to

$$x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_k}$$

where  $n = n_1 + \dots + n_k$ .

Consider the **separable** optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^k f_i(x_i) \\ \text{subject to} \quad & \sum_{i=1}^k g_{ij}(x_i) \leq 0, \quad \forall 1 \leq j \leq r \\ & x_i \in \Omega_i, \quad \forall 1 \leq i \leq k \end{aligned}$$

Here,

$$f_i : \Omega_i \rightarrow \mathbb{R}, \quad g_{ij} : \Omega_i \rightarrow \mathbb{R}, \quad \Omega_i \subset \mathbb{R}^{n_i}$$

## Separability and Duality

---

The Lagrangian is

$$L(x, \mu) = \sum_{i=1}^k f_i(x_i) + \sum_{j=1}^r \mu_j \sum_{i=1}^k g_{ij}(x_i)$$

The dual function is

$$q(\mu) = \inf_{x \in \Omega_1 \times \dots \times \Omega_k} \sum_{i=1}^k \left( f_i(x_i) + \sum_{j=1}^r \mu_j g_{ij}(x_i) \right)$$

Note that if

$$q_i(\mu) \triangleq \inf_{x_i \in \Omega_i} f_i(x_i) + \sum_{j=1}^r \mu_j g_{ij}(x_i)$$

then the dual problem becomes

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^k q_i(\mu) \\ \text{subject to} \quad & \mu \geq 0 \\ & \mu \in \mathbb{R}^r \end{aligned}$$

## Example: Resource Allocation

---

- Activities  $1, \dots, k$  (e.g., divisions of a firm)
- Resources  $1, \dots, r$  (e.g., capital, labor, etc.)
- Each activity consumes resources, and generates a benefit (utility, profit, etc.)
- Decision variables:

$x_{ij}$  = quantity of resource  $j$  allocated to activity  $i$

$$x_{ij} \geq 0$$

- The  $i$ th activity generates utility according to

$$U_i(x_i) \triangleq U_i(x_{i1}, \dots, x_{ir})$$

- The supply of the resources is limited, so we require that

$$\sum_{i=1}^k x_{ij} \leq C_j, \quad \forall 1 \leq j \leq r \quad [C_j > 0]$$

## Example: Resource Allocation

---

- Objective: maximize total utility

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^k U_i(x_i) \\ &\text{subject to} && \sum_{i=1}^k x_{ij} \leq C_j, \quad \forall 1 \leq j \leq r \\ &&& x \geq 0, \\ &&& x \in \mathbb{R}^{k \times r} \end{aligned}$$

## Example: Resource Allocation

---

Lagrangian:

$$L(x, \mu) = \sum_{i=1}^k U_i(x_i) - \sum_{j=1}^r \mu_j \left( \sum_{i=1}^k x_{ij} - C_j \right)$$

Dual function:

$$q(\mu) = \sup_{x \geq 0} L(x, \mu) = \sum_{i=1}^k q_i(\mu) + \sum_{j=1}^r \mu_j C_j$$

where

$$q_i(\mu) = \sup_{x_i \geq 0} U_i(x_i) - \sum_{j=1}^r \mu_j x_{ij}$$

## Example: Resource Allocation

---

If  $\mu^*$  is a geometric multiplier, the a feasible allocation  $x^*$  is a global minimum if and only if

$$x^* \in \operatorname{argmax}_{x \geq 0} L(x, \mu^*) = \operatorname{argmax}_{x \geq 0} \sum_{i=1}^k \left( U_i(x_i) - \sum_{j=1}^r \mu_j^* x_{ij} \right)$$

$$\Leftrightarrow x_i^* \in \operatorname{argmax}_{x_i \geq 0} U_i(x_i) - \sum_{j=1}^r \mu_j^* x_{ij}$$

The dual variables  $\mu^*$  can be interpreted as **prices** that serve as a **coordination mechanism**. Given the proper selection of prices, the optimal solution can be constructed by solving **independent** subproblems for each activity.

Prices are **proxies for decentralization**.

## Example: Resource Allocation

Conditions for optimal solution-geometric multiplier pair  $(x^*, \mu^*)$ :

(i) Primal feasibility

$$x^* \geq 0, \quad \sum_{i=1}^k x_{ij}^* \leq C_j, \quad \forall 1 \leq j \leq r$$

(ii) Dual feasibility  $\mu^* \geq 0$

(iii) Lagrangian optimality

$$x_i^* \in \operatorname{argmax}_{x_i \geq 0} U_i(x_i) - \sum_{j=1}^r \mu_j^* x_{ij}$$

(iv) Complementary slackness

$$\mu_j^* \left( \sum_{i=1}^k x_{ij}^* - C_j \right) = 0, \quad \forall 1 \leq j \leq r$$

## Tâtonnement Procedure

1. Pick a set of initial prices  $\mu \geq 0$ .
2. Compute the allocation  $x$  by

$$x_i \in \operatorname{argmax}_{x'_i \geq 0} U_i(x'_i) - \sum_{j=1}^r \mu_j x'_{ij}$$

3. For each resource  $j$ ,

$$\text{if } \sum_{i=1}^k x_{ij} > C_j \quad \Rightarrow \quad \text{raise } \mu_j \text{ slightly}$$

$$\text{if } \sum_{i=1}^k x_{ij} < C_j \quad \Rightarrow \quad \text{lower the } \mu_j \text{ slightly, keeping } \mu_j \geq 0$$

4. Repeat.

Under proper technical conditions (concavity of utility functions, existence of geometric multipliers, good choice of step-sizes, etc.), this decentralized procedure will find the global minimum.

The **knapsack problem** consists of determining which of  $n$  items to place in a knapsack of limited capacity.

- Decision variables:  $(x_1, \dots, x_n)$  where  $x_i = 1$  if the item  $i$  is placed in the knapsack,  $x_i = 0$  otherwise
- Knapsack constraint: the total capacity of the knapsack is  $C > 0$ , item  $i$  has weight  $0 < w_i \leq C$ , so

$$\sum_{i=1}^n w_i x_i \leq C$$

- Objective: the value of item  $i$  is  $v_i > 0$ , we would like to maximize the total value of items in the knapsack

$$\sum_{i=1}^n v_i x_i$$

## Example: The Knapsack Problem

$$\begin{aligned} f^* = \text{maximize} \quad & f(x) = \sum_{i=1}^n v_i x_i \\ \text{subject to} \quad & \sum_{i=1}^n w_i x_i \leq C \\ & x_i \in \{0, 1\}, \quad \forall 1 \leq i \leq n \end{aligned}$$

Integer problem, “hard”

Dual function:

$$\begin{aligned} q(\mu) &= \sup_{x \in \{0,1\}^n} L(x, \mu) = \sup_{x \in \{0,1\}^n} \sum_{i=1}^n v_i x_i - \mu \left( \sum_{i=1}^n w_i x_i - C \right) \\ &= \sup_{x \in \{0,1\}^n} \sum_{i=1}^n (v_i - \mu w_i) x_i + \mu C \\ &= \sum_{i=1}^n \max(v_i - \mu w_i, 0) + \mu C \end{aligned}$$

## Example: The Knapsack Problem

---

Weak duality:

$$f^* \leq q^* = \inf_{\mu \geq 0} q(\mu) = \inf_{\mu \geq 0} \sum_{i=1}^n w_i \max(v_i/w_i - \mu, 0) + \mu C$$

Without loss of generality, assume

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \dots \geq \frac{v_n}{w_n}$$

Define

$$I^*(\mu) \triangleq \max \{i : v_i/w_i \geq \mu\}$$

Then,

$$q(\mu) = \sum_{i=1}^{I^*(\mu)} (v_i - w_i \mu) + \mu C = \sum_{i=1}^{I^*(\mu)} v_i + \mu \left( C - \sum_{i=1}^{I^*(\mu)} w_i \right) \\ \Rightarrow \text{piecewise linear}$$

## Example: The Knapsack Problem

---

Define

$$I^* \triangleq \min \left\{ I : \sum_{i=1}^I w_i > C \right\} \Rightarrow \mu^* = v_{I^*}/w_{I^*}$$

Then,

$$f^* \leq q^* = \inf_{\mu \geq 0} q(\mu) = \sum_{i=1}^{I^*} v_i + \frac{v_{I^*}}{w_{I^*}} \left( C - \sum_{i=1}^{I^*} w_i \right) \\ = \sum_{i=1}^{I^*-1} v_i + \frac{v_{I^*}}{w_{I^*}} \left( C - \sum_{i=1}^{I^*-1} w_i \right)$$

We have an upper bound on the optimal value. How about a “good” solution? A bound on the duality gap?

## Example: The Knapsack Problem

---

Consider the optimization

$$\max_{x \in \{0,1\}^n} L(x, \mu^*) = \max_{x \in \{0,1\}^n} \sum_{i=1}^n \left( v_i - \frac{v_{I^*}}{w_{I^*}} w_i \right) x_i + \frac{v_{I^*}}{w_{I^*}} C$$
$$\Rightarrow \tilde{x}_i = \begin{cases} 1 & \text{if } i < I^* \\ 0 & \text{if } i \geq I^* \end{cases}$$

Note that the trial solution  $\tilde{x}$  is clearly feasible. It is constructed with a **greedy** algorithm: sort the items by “bang per buck”  $v_i/w_i$ , greedily add items to knapsack until it is full.

$$f(\tilde{x}) = \sum_{i=1}^{I^*-1} v_i \quad q^* - f(\tilde{x}) = \frac{v_{I^*}}{w_{I^*}} \left( C - \sum_{i=1}^{I^*-1} w_i \right) \leq v_{I^*}$$

## Example: The Knapsack Problem

---

Putting it all together,

$$f(\tilde{x}) \leq f^* \leq f(\tilde{x}) + v_{I^*}$$

Can we do better?

Set  $\hat{x}$  to be the the better of

1. the trial solution  $\tilde{x}$
2. a solution consisting of only item  $I^*$

$$f(\hat{x}) \leq f^* \leq f(\tilde{x}) + v_{I^*} \leq 2f(\hat{x})$$

The revised algorithm is a **2-approximation** to the knapsack problem.

We would like to understand two related questions:

- When is there no duality gap?
- When do optimal solutions exist for the dual problem?

## Strong Duality: Linear Constraints

---

Primal problem:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$
$$A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax \leq b, \\ & x \in \mathbb{R}^n \end{array}$$

Dual problem:

$$L(x, \mu) = f(x) + \mu^\top (Ax - b)$$

$$q(\mu) = \inf_{x \in \mathbb{R}^n} L(x, \mu)$$

$$\begin{array}{ll} \text{maximize} & q(\mu) \\ \text{subject to} & \mu \geq 0, \\ & \mu \in \mathbb{R}^m \end{array}$$

**Theorem.** Suppose that  $f(\cdot)$  is **convex** over  $\mathbb{R}^n$  and **continuously differentiable**. If the primal problem has an optimal solution, then there is no duality gap, and at least one geometric multiplier exists.

Let  $x^*$  be an optimal solution for the primal. Then, there exists  $\mu^* \in \mathbb{R}^m$  such that

$$\mu^* \geq 0, \quad (\mu^*)^\top (Ax^* - b) = 0, \quad \nabla f(x^*) + A\mu^* = 0$$

Since  $L(x, \mu)$  is convex in  $x$ , we have

$$x^* \in \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} L(x, \mu^*)$$

Thus,

$$f(x^*) = \min_{x \in \mathbb{R}^n} f(x) + (\mu^*)^\top (Ax - b) = q(\mu^*)$$

Then,

$$q(\mu^*) \leq q^* \leq f^* \leq f(x^*) \quad \Rightarrow \quad q^* = f^*$$

## Linear Constraints: Remarks

---

Trivially applies to linear equality constraints also. More generally,

$$\begin{array}{ll} f : \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \mathbb{R}^n \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0, \\ g : \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0, \\ & x \in \mathbb{R}^n \end{array}$$

$$L(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

If:

- There exists an optimal solution  $x^*$
- $f(\cdot)$ ,  $h(\cdot)$ ,  $g(\cdot)$  are continuously differentiable
- There exist multipliers  $(\lambda^*, \mu^*)$  satisfying the KKT conditions (regularity)
- $f(\cdot)$ ,  $h(\cdot)$ ,  $g(\cdot)$  are convex over  $\mathbb{R}^n$

Then, there is no duality gap, and geometric multipliers exist.

We would like a more general theory, which

- Does not require differentiability
- Allows for set constraints

## Slater's Condition

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Theorem.** (Slater's Condition) Suppose that:

- (i) The problem is bounded, i.e.

$$-\infty < f^* = \inf_{x \in \Omega, g(x) \leq 0} f(x)$$

- (ii) The set  $\Omega$  is convex, and  $f(\cdot)$ ,  $g(\cdot)$  are convex over  $\Omega$

- (iii) There exists a vector  $\bar{x} \in \Omega$  with  $g(\bar{x}) < 0$

Then, there is no duality gap and there exists at least one geometric multiplier.

## Slater's Condition: Proof

---

Define

$$\mathcal{A} = \{(z, w) \in \mathbb{R}^{r+1} : \exists x \in \Omega \text{ with } g(x) \leq z, f(x) \leq w\}$$

Observe that  $\mathcal{A}$  is convex, by the convexity of  $\Omega$ ,  $f(\cdot)$ , and  $g(\cdot)$ .

Next, observe that  $(0, f^*)$  is not in the interior of  $\mathcal{A}$ . Otherwise, for some  $\epsilon > 0$ ,  $(0, f^* - \epsilon) \in \mathcal{A}$ , contradicting the definition of  $f^*$ .

By the supporting hyperplane theorem, there exists a normal vector  $(\mu, \beta) \neq (0, 0)$  such that

$$\beta f^* \leq \beta w + \mu^\top z, \quad \forall (z, w) \in \mathcal{A}$$

If  $(z, w) \in \mathcal{A}$ , then  $(z, w + \gamma) \in \mathcal{A}$  for all  $\gamma \geq 0$ . Then, we must have  $\beta \geq 0$ . Similarly,  $\mu \geq 0$ .

## Slater's Condition: Proof

---

We would like to show that  $\beta > 0$ . Suppose not. Then,

$$0 \leq \mu^\top z, \quad \forall (z, w) \in \mathcal{A}$$

Since  $(g(\bar{x}), f(\bar{x})) \in \mathcal{A}$ ,

$$0 \leq \mu^\top g(\bar{x})$$

Then, we must have  $\mu = 0$  and  $(\mu, \beta) = (0, 0)$  which is a contradiction.

Since  $\beta > 0$ , we can divide by  $\beta$  and assume that  $\beta = 1$ . Thus,

$$f^* \leq w + \mu^\top z, \quad \forall (z, w) \in \mathcal{A}$$

$$\Rightarrow f^* \leq f(x) + \mu^\top g(x), \quad \forall x \in \Omega$$

Minimizing over  $x \in \Omega$ ,

$$f^* \leq \inf_{x \in \Omega} f(x) + \mu^\top g(x) = q(\mu) \leq q^*$$

Then, by weak duality,  $f^* = q^*$  and  $\mu$  is a geometric multiplier.

Existence of the interior point in condition (iii) is important!

Does not apply to equality constraints! These are harder to deal with.

**Definition.** Suppose  $\mathcal{C} \subset \mathbb{R}^n$  is a convex set. The **relative interior** of  $\mathcal{C}$  is the set **relint**  $\mathcal{C}$  of all  $x \in \mathbb{R}^n$  for which there exists an  $\epsilon > 0$  such that if  $z \in \mathbf{aff} \mathcal{C}$  with  $\|z - x\| < \epsilon$ , then  $z \in \mathcal{C}$ .

In other words, **relint**  $\mathcal{C}$  is the interior of  $\mathcal{C}$  relative to the affine hull **aff**  $\mathcal{C}$ .

## Mixed Constraints

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq 0 \\ A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m & Ax \leq b \\ E \in \mathbb{R}^{m \times k}, d \in \mathbb{R}^k & Ex = d \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Theorem.** Suppose that the optimal value  $f^*$  is finite, and:

- (i)  $\Omega$  is the intersection of a convex set  $\mathcal{C}$  and a polyhedron
- (ii) The functions  $f(\cdot)$ ,  $g(\cdot)$  are convex over  $\Omega$
- (iii) There is a feasible vector  $\bar{x}$  with  $g(\bar{x}) < 0$
- (iv) There is a vector  $x$  with  $Ax \leq b$ ,  $Ex = d$ ,  $x \in \mathbf{relint} \mathcal{C}$ , and  $x \in \Omega$

Then, there is no duality gap and there exists at least one geometric multiplier.

B9824 Foundations of Optimization

## Lecture 7: Duality III

Fall 2008

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### Outline

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1. Duality for linear programs, quadratic programs
2. Conjugacy & duality
3. Chernoff bounds & large deviations
4. Sensitivity

## Duality for Linear Programs

$$\begin{array}{ll}
 c \in \mathbb{R}^n & \text{minimize} \quad c^\top x \\
 A \in \mathbb{R}^{r \times n} & \text{subject to} \quad Ax \geq b \\
 b \in \mathbb{R}^r & x \in \Omega = \mathbb{R}^n
 \end{array}$$

The Lagrangian is

$$L(x, \mu) = c^\top x + \mu^\top (b - Ax)$$

The dual function is

$$q(\mu) = \inf_x c^\top x + \mu^\top (b - Ax) = \begin{cases} b^\top \mu & \text{if } c = A^\top \mu \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem is

$$\begin{array}{ll}
 \text{maximize} & q(\mu) \\
 \text{subject to} & \mu \geq 0 \\
 & \mu \in \mathbb{R}^r
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{ll}
 \text{maximize} & b^\top \mu \\
 \text{subject to} & A^\top \mu = c \\
 & \mu \geq 0 \\
 & \mu \in \mathbb{R}^r
 \end{array}$$

## Duality for Linear Programs

More generally, let  $A$  be a matrix with rows  $a_i \in \mathbb{R}^n$  and columns  $A_j \in \mathbb{R}^n$ . Then:

	<b>Primal</b>		<b>Dual</b>
minimize	$c^\top x$	maximize	$b^\top y$
subject to	$a_i^\top x \geq b_i, \quad \forall i \in M_1$	subject to	$y_j \geq 0, \quad \forall j \in M_1$
	$a_i^\top x \leq b_i, \quad \forall i \in M_2$		$y_j \leq 0, \quad \forall j \in M_2$
	$a_i^\top x = b_i, \quad \forall i \in M_3$		$y_j$ <b>free</b> , $\forall j \in M_3$
	$x_j \geq 0, \quad \forall j \in N_1$		$A_j^\top y \leq c_j, \quad \forall j \in N_1$
	$x_j \leq 0, \quad \forall j \in N_2$		$A_j^\top y \geq c_j, \quad \forall j \in N_2$
	$x_j$ <b>free</b> , $\forall j \in N_3$		$A_j^\top y = c_j, \quad \forall j \in N_3$

**Note:** The dual is being taken with respect to the set constraint

$$\Omega \triangleq \{x \in \mathbb{R}^n : x_j \geq 0, \forall j \in N_1, x_j \leq 0, \forall j \in N_2\}$$

<b>Primal</b>	minimize	maximize	<b>Dual</b>
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	$\geq 0$ $\leq 0$ <b>free</b>	variables
variables	$\geq 0$ $\leq 0$ <b>free</b>	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

## Duality for Quadratic Programs

$$\begin{aligned}
 Q &\in \mathbb{R}^{n \times n}, Q \succ 0 \\
 c &\in \mathbb{R}^n \\
 A &\in \mathbb{R}^{r \times n} \\
 b &\in \mathbb{R}^r
 \end{aligned}$$

$$\begin{aligned}
 &\text{minimize} && \frac{1}{2}x^\top Qx + c^\top x \\
 &\text{subject to} && Ax \leq b \\
 &&& x \in \Omega = \mathbb{R}^n
 \end{aligned}$$

The Lagrangian is

$$L(x, \mu) = \frac{1}{2}x^\top Qx + c^\top x + \mu^\top (Ax - b)$$

The dual function is

$$q(\mu) = \inf_x \frac{1}{2}x^\top Qx + c^\top x + \mu^\top (Ax - b)$$

This is minimized when

$$x = -Q^{-1}(c + A^\top \mu)$$

Thus,

$$q(\mu) = -\frac{1}{2}\mu^\top A Q^{-1} A^\top \mu - \mu^\top (b + A Q^{-1} c) - \frac{1}{2}c^\top Q^{-1} c$$

## Duality for Quadratic Programs

$$\begin{aligned} Q &\in \mathbb{R}^{n \times n}, Q \succ 0 \\ c &\in \mathbb{R}^n \\ A &\in \mathbb{R}^{r \times n} \\ b &\in \mathbb{R}^r \end{aligned}$$

$$\begin{aligned} &\text{minimize} && \frac{1}{2}x^\top Qx + c^\top x \\ &\text{subject to} && Ax \leq b \\ &&& x \in \Omega = \mathbb{R}^n \end{aligned}$$

The dual problem is

$$\begin{aligned} &\text{maximize} && q(\mu) \\ &\text{subject to} && \mu \geq 0 \\ &&& \mu \in \mathbb{R}^r \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\text{maximize} && -\frac{1}{2}\mu^\top P\mu - t^\top \mu - d \\ &\text{subject to} && \mu \geq 0 \\ &&& \mu \in \mathbb{R}^r \end{aligned}$$

Here,

$$P \triangleq AQ^{-1}A^\top, \quad t \triangleq b + AQ^{-1}c, \quad d \triangleq \frac{1}{2}c^\top Q^{-1}c$$

**Note:** The dual has simpler constraints and possibly smaller dimension. However, dual may be dense if primal is sparse.

## Conjugate Functions

**Definition.** A **proper** convex function is an extended-real valued convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ , with **dom**  $f \neq \emptyset$ .

**Definition.** A **closed** function is an extended-real valued function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ , such that for every  $\alpha \in \mathbb{R}$ , the sublevel set

$$\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

is closed.

**Definition.** The **convex conjugate** (Fenchel-Legendre transformation) of a proper convex function  $f(\cdot)$  is the extended-real valued function  $f^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$f^*(y) \triangleq \sup_{x \in \text{dom } f} x^\top y - f(x)$$

**Example.**

$$f(x) = a^\top x + b \quad \Rightarrow \quad f^*(y) = \begin{cases} -b & \text{if } y = a \\ \infty & \text{otherwise} \end{cases}$$

**Example.**

$$f(x) = \frac{1}{2}x^\top Qx, \quad Q \succ 0 \quad \Rightarrow \quad f^*(y) = \frac{1}{2}y^\top Q^{-1}y$$

**Example.**

$$f(x) = \log \left( \sum_{i=1}^n e^{x_i} \right) \quad \Rightarrow \quad f^*(y) = \begin{cases} \sum_{i=1}^n y_i \log y_i & \text{if } y \geq 0, \mathbf{1}^\top y = 1 \\ \infty & \text{otherwise} \end{cases}$$

## Elementary Properties of Conjugate Functions

---

**Theorem.**  $f^*(\cdot)$  is convex.

**Proof.** Follows since it is a pointwise supremum of convex (linear) functions in  $y$ . This is true even if  $f(\cdot)$  is not convex.  $\square$

**Theorem.** (Fenchel's Inequality) For all  $x, y \in \mathbb{R}^n$ ,

$$x^\top y \leq f(x) + f^*(y)$$

**Proof.** Follows from the definition, since

$$f^*(y) \geq x^\top y - f(x)$$

$\square$

**Theorem.** If  $f(u, v) = f_1(u) + f_2(v)$ , then

$$f^*(w, z) = f_1^*(w) + f_2^*(z)$$

**Proof.** Follows from the definition.  $\square$

**Theorem.** If  $f(\cdot)$  is a proper, closed convex function, then  $f^{**} = f$ .

## Differentiability

---

**Theorem.** Suppose  $f(\cdot)$  is a convex and differentiable function with  $\text{dom } f = \mathbb{R}^n$ . Then, if  $x \in \mathbb{R}^n$  and  $y = \nabla f(x)$ , then

$$f^*(y) = x^\top \nabla f(x) - f(x)$$

**Proof.** If  $y = \nabla f(x)$ , then  $z = x$  maximizes the function  $z^\top y - f(z)$  (note that this function is concave and differentiable, so first order conditions are sufficient).  $\square$

**Note:** This allows us to determine  $f^*(y)$  for any  $y$  where we can solve  $y = \nabla f(x)$  for  $x$ .

## Example: Chernoff Bounds & Large Deviations

---

Suppose that  $X$  is a real-valued random variable. If  $X \geq 0$ , and  $t > 0$  is a constant,

$$tI_{\{X \geq t\}} \leq X \Rightarrow t\mathbf{E}[I_{\{X \geq t\}}] \leq \mathbf{E}[X] \Rightarrow \Pr(X \geq t) \leq \frac{1}{t}\mathbf{E}[X]$$

The last inequality is known as **Markov's inequality**.

Now, if  $X$  is any real-valued random variable and  $\lambda > 0$  is a constant,

$$\Pr(X \geq t) = \Pr(\lambda X \geq \lambda t) = \Pr(e^{\lambda X} \geq e^{\lambda t}) \leq e^{-\lambda t} \mathbf{E}[e^{\lambda X}]$$

This is known as a **Chernoff** bound. The inequality trivially holds for  $\lambda = 0$ , so we have

$$\Pr(X \geq t) \leq e^{-\lambda t} \mathbf{E}[e^{\lambda X}], \quad \forall \lambda \geq 0$$

## Example: Chernoff Bounds & Large Deviations

---

Define the **cumulant generating function**

$$f(\lambda) \triangleq \log \mathbf{E}[e^{\lambda X}]$$

Note that this is always a convex function!

Optimizing over the choice of  $\lambda \geq 0$  in the Chernoff bound,

$$\Pr(X \geq t) \leq \inf_{\lambda \geq 0} e^{-\lambda t} \mathbf{E}[e^{\lambda X}] = \exp\left(-\sup_{\lambda \geq 0} \lambda t - f(\lambda)\right)$$

The tightest possible bound is closely related to the conjugate of  $f(\cdot)$ !

## Example: Chernoff Bounds & Large Deviations

---

Consider a collection of random vectors  $\{X_1, X_2, \dots\}$ , where each  $X_i \in \{0, 1\}$  is independently and identically distributed coin flip, and  $E[X_i] = 1/2$ .

Consider the cumulative ‘number of heads’

$$Y_k = \sum_{i=1}^k X_i$$

By the law of large numbers, we expect that  $Y_k/k \approx 1/2$ . But how fast? To be precise, if  $\alpha > 1/2$ , how fast does

$$\Pr(Y_k/k \geq \alpha)$$

go to zero? The study of this question is known as **large deviations**.

## Example: Chernoff Bounds & Large Deviations

---

Note that, if  $\lambda \geq 0$ ,

$$\Pr(Y_k \geq \alpha k) \leq e^{-\alpha \lambda k} \mathbf{E} \left[ e^{\lambda Y_k} \right] = e^{-\alpha \lambda k} \left( \mathbf{E} \left[ e^{\lambda X_1} \right] \right)^k$$

Define

$$f(\lambda) \triangleq \log \mathbf{E} \left[ e^{\lambda X_1} \right] = \log (1 + e^\lambda) - \log 2$$

Then,

$$\Pr(Y_k \geq \alpha k) \leq \exp \{ -k (\lambda \alpha - f(\lambda)) \}$$

When  $1/2 < \alpha < 1$ ,

$$\Pr(Y_k \geq \alpha k) \leq \exp \left( -k \sup_{\lambda \geq 0} \lambda \alpha - f(\lambda) \right) = \exp (-k f^*(\alpha))$$

and the probability goes to zero exponentially fast at the rate

$$f^*(\alpha) = \alpha \log \alpha + (1 - \alpha) \log(1 - \alpha) + \log 2$$

## Conjugacy & Duality

---

Consider a proper convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ . Consider the optimization problem

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && x = 0 \\ & && [x \in \mathbf{dom} f] \end{aligned}$$

The Lagrangian is

$$L(x, \lambda) = f(x) + \lambda^\top x$$

The dual function is

$$q(\lambda) = \inf_x f(x) + \lambda^\top x = -\sup_x (-\lambda)^\top x - f(x) = -f^*(-\lambda)$$

## Conjugacy & Duality

---

$f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$   
proper, convex  
 $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$   
 $C \in \mathbb{R}^{r \times n}$ ,  $d \in \mathbb{R}^r$

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && Ax = b \\ & && Cx \leq d \\ & && [x \in \mathbf{dom} f] \end{aligned}$$

The Lagrangian is

$$L(x, \lambda, \mu) = f(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d)$$

The dual function is

$$\begin{aligned} q(\lambda, \mu) &= \inf_x f(x) + \lambda^\top (Ax - b) + \mu^\top (Cx - d) \\ &= -b^\top \lambda - d^\top \mu + \inf_x f(x) + \lambda^\top Ax + \mu^\top Cx \\ &= -b^\top \lambda - d^\top \mu - f^*(-A^\top \lambda - C^\top \mu) \end{aligned}$$

In general, in order of difficulty:

- Lookup a standard form, e.g., linear program
- Use conjugacy
- Directly from the definition

In general, the dual is sensitive to problem formulation. For example, the problems

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = b \\ & x \in \mathbb{R}^n \end{array} \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}\|x\|^2 \\ \text{subject to} & Ax = b \\ & x \in \mathbb{R}^n \end{array}$$

are mathematically equivalent, but have very different duals!

## The Primal Problem

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize} \quad f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to} \quad g(x) \leq u \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Definition.** The **primal function** is given by

$$p(u) \triangleq \inf_{x \in \Omega, g(x) \leq u} f(x)$$

with domain

$$\mathbf{dom} \, p \triangleq \{u \in \mathbb{R}^r : \exists x \in \Omega \text{ with } g(x) \leq u\}$$

In order to keep things simple, we will make the assumption that the primal problem is always bounded.

**Assumption.** Assume that  $p(u) > -\infty$  for all  $u \in \mathbf{dom} \, p$ . Then,  $p : \mathbb{R}^r \rightarrow \mathbb{R} \cup \{\infty\}$  is an extended-real valued function.

## The Primal Problem

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq u \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Theorem.** Suppose that  $\Omega$  is convex, and that  $f(\cdot)$  and  $g_j(\cdot)$ ,  $1 \leq j \leq r$ , are convex over  $\Omega$ . Then,  $p(\cdot)$  is convex.

**Theorem.** If  $u_1 \geq u_2$ , then  $p(u_1) \leq p(u_2)$ .

## The Dual Function

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq u \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

Then, for  $\mu \geq 0$ ,

$$\begin{aligned} q(\mu) &\triangleq \inf_{x \in \Omega} f(x) + \mu^\top g(x) \\ &= \inf_{x, u} \left\{ f(x) + \mu^\top g(x) : x \in \Omega, u \in \mathbb{R}^r, g(x) \leq u \right\} \\ &= \inf_{x, u} \left\{ f(x) + \mu^\top u : x \in \Omega, u \in \mathbb{R}^r, g(x) \leq u \right\} \\ &= \inf_{u \in \mathbb{R}^r} \inf_{x \in \Omega, g(x) \leq u} f(x) + \mu^\top u \\ &= \inf_{u \in \mathbb{R}^r} \mu^\top u + \inf_{x \in \Omega, g(x) \leq u} f(x) \\ &= -p^*(-\mu) \end{aligned}$$

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ g : \Omega \rightarrow \mathbb{R}^r & \text{subject to } g(x) \leq u \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

**Theorem.** Suppose that strong duality holds, and the dual optimum is attained when  $u = 0$ , with  $\mu^*$  being a geometric multiplier. Then, for all  $u \in \mathbb{R}^n$ ,

$$p(u) \geq p(0) - u^\top \mu^*$$

**Proof.** For all  $x \in \Omega$  with  $g(x) \leq u$ ,

$$p(0) = q(\mu^*) \leq f(x) + g(x)^\top \mu^* \leq f(x) + u^\top \mu^*$$

The result follows. □

**Note:** If  $p(\cdot)$  is convex and differentiable, this result implies that

$$\nabla p(0) = -\mu^*$$

B9824 Foundations of Optimization

## Lecture 8: Problem Formulation

Fall 2008

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### Outline

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1. Standard forms of convex programs
2. Equivalent formulations
3. Geometric programming
4. Robust optimization
5. Parameter estimation

# Why is Convexity Important?

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- A convex function has no local minima that are not global minima
- A convex set is connected and has feasible directions at every point
- A convex set can be characterized by extreme points or supporting hyperplanes
- Efficient algorithms are available for solving convex optimization problems

## Equivalent Formulations

---

The same optimization problem can admit multiple, different formulations. Some formulations may be more advantageous than others:

- More or fewer variables/constraints
- Simpler or more complicated objective function/constraint set
- Sparse vs. not-sparse
- Convex vs. non-convex
- Differentiable vs. non-differentiable
- Decentralized vs. centralized
- Different dual problems

## Standard Forms

---

If an optimization problem can be converted to a **standard form**, it can be solved efficiently by off-the-shelf software.

Some common standard forms for convex problems are:

### Linear Program (LP)

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & Gx \leq h \\ & x \in \mathbb{R}^n \end{array} \quad \begin{array}{l} c \in \mathbb{R}^n \\ A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\ G \in \mathbb{R}^{r \times n}, h \in \mathbb{R}^r \end{array}$$

linear objective, linear equality/inequality constraints

## Standard Forms

---

### Quadratic Program (QP)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top Qx + c^\top x \\ \text{subject to} & Ax = b \\ & Gx \leq h \\ & x \in \mathbb{R}^n \end{array} \quad \begin{array}{l} Q \in \mathbb{R}^{n \times n} \text{ symmetric, positive} \\ \text{semidefinite} \\ c \in \mathbb{R}^n \\ A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\ G \in \mathbb{R}^{r \times n}, h \in \mathbb{R}^r \end{array}$$

convex quadratic objective, linear equality/inequality constraints

Examples: portfolio optimization, linear regression

LP  $\subset$  QP

### Quadratically Constrained Quadratic Program (QCQP)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^\top Qx + c^\top x \\ \text{subject to} & Ax = b \\ & \frac{1}{2}x^\top P_i x + g_i^\top x + h_i \leq 0, \\ & \forall 1 \leq i \leq r \\ & x \in \mathbb{R}^n \end{array} \quad \begin{array}{l} Q, P_i \in \mathbb{R}^{n \times n} \text{ symmetric,} \\ \text{positive semidefinite} \\ c \in \mathbb{R}^n \\ A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\ g_i \in \mathbb{R}^n, h_i \in \mathbb{R} \end{array}$$

convex quadratic objective, convex quadratic inequality constraints,  
linear equality constraints

$$\text{LP} \subset \text{QP} \subset \text{QCQP}$$

### Second Order Cone Program (SOCP)

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & \|F_i x + q_i\| \leq g_i^\top x + h_i, \\ & \forall 1 \leq i \leq r \\ & x \in \mathbb{R}^n \end{array} \quad \begin{array}{l} c \in \mathbb{R}^n \\ A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \\ F_i \in \mathbb{R}^{n_i \times n}, q_i \in \mathbb{R}^{n_i} \\ g_i \in \mathbb{R}^n, h_i \in \mathbb{R} \end{array}$$

linear objective and equality constraints, second-order cone  
constraints

$$\text{LP} \subset \text{QP} \subset \text{QCQP} \subset \text{SOCP}$$

## Semidefinite Program (SDP)

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x_1 F_1 + \cdots + x_n F_n \preceq 0 \\ & x \in \mathbb{R}^n \end{array} \quad \begin{array}{l} F_i \in \mathbb{R}^{k \times k}, \text{ symmetric} \\ c \in \mathbb{R}^n \\ A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \end{array}$$

linear objective and equality constraints, linear matrix inequalities

$$\text{LP} \subset \text{QP} \subset \text{QCQP} \subset \text{SOCP} \subset \text{SDP}$$

## Transformation of Objective/Constraints

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize} \quad f(x) \\ h : \Omega \rightarrow \mathbb{R}^m & \text{subject to} \quad h(x) = 0 \\ g : \Omega \rightarrow \mathbb{R}^r & \quad g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & \quad x \in \Omega \end{array}$$

Suppose:

- $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing function
- $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  satisfies  $\psi(u) = 0$  if and only if  $u = 0$
- $\chi : \mathbb{R}^r \rightarrow \mathbb{R}^r$  satisfies  $\chi(v) \leq 0$  if and only if  $v \leq 0$

Then, an equivalent problem is:

$$\begin{array}{ll} \tilde{f}(x) = \phi(f(x)) & \text{minimize} \quad \tilde{f}(x) \\ \tilde{h}(x) = \psi(h(x)) & \text{subject to} \quad \tilde{h}(x) = 0 \\ \tilde{g}(x) = \chi(g(x)) & \quad \tilde{g}(x) \leq 0 \\ & \quad x \in \Omega \end{array}$$

## Transformation of Objective/Constraints: Example

---

Consider the least-norm approximation problems:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\| \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

These problems have the same global optimum, and both are convex. However:

- the objective in first problem is not differentiable for  $x$  with  $Ax - b = 0$ , it is an SOCP
- the second objective is differentiable for all  $x$ , and, in fact, is a QP

## Change of Variables

---

$$\begin{array}{l} f : \Omega \rightarrow \mathbb{R} \\ h : \Omega \rightarrow \mathbb{R}^m \\ g : \Omega \rightarrow \mathbb{R}^r \\ \Omega \subset \mathbb{R}^n \end{array}$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) = 0 \\ & g(x) \leq 0 \\ & x \in \Omega \end{array}$$

Suppose  $\phi : \tilde{\Omega} \rightarrow \Omega$  is **surjective**. Then, an equivalent problem is:

$$\begin{array}{l} \tilde{f}(z) = f(\phi(z)) \\ \tilde{h}(z) = h(\phi(z)) \\ \tilde{g}(z) = g(\phi(z)) \end{array}$$

$$\begin{array}{ll} \text{minimize} & \tilde{f}(z) \\ \text{subject to} & \tilde{h}(z) = 0 \\ & \tilde{g}(z) \leq 0 \\ & z \in \tilde{\Omega} \end{array}$$

## Example: Geometric Programming

---

**Definition.** A **monomial** is a function  $f : (0, \infty)^n \rightarrow \mathbb{R}$  of the form

$$f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

where  $c > 0$  and  $a_i \in \mathbb{R}$ .

**Definition.** A **posynomial** is a function  $f : (0, \infty)^n \rightarrow \mathbb{R}$  that is a sum of monomials, that is, of the form

$$f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

where  $c_k > 0$  and  $a_{ik} \in \mathbb{R}$ .

## Example: Geometric Programming

---

**Definition.** A **geometric program** (GP) is an optimization program of the form

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && h_i(x) = 1, \quad \forall 1 \leq i \leq m \\ & && g_j(x) \leq 1, \quad \forall 1 \leq j \leq r \\ & && x > 0 \\ & && x \in \mathbb{R}^n \end{aligned}$$

where

- $f : (0, \infty)^n \rightarrow \mathbb{R}$  is a posynomial
- each  $h_i : (0, \infty)^n \rightarrow \mathbb{R}$  is a monomial
- each  $g_j : (0, \infty)^n \rightarrow \mathbb{R}$  is a posynomial

## Example: Maximum Volume Box

---

$$\begin{aligned} &\text{maximize} && x_1 x_2 x_3 \\ &\text{subject to} && x_1 x_2 + x_2 x_3 + x_1 x_3 \leq c/2 \quad (c > 0) \\ &&& x > 0 \\ &&& x \in \mathbb{R}^3 \end{aligned}$$

This is equivalent to the standard form GP

$$\begin{aligned} &\text{minimize} && x_1^{-1} x_2^{-1} x_3^{-1} \\ &\text{subject to} && \frac{2}{c} x_1 x_2 + \frac{2}{c} x_2 x_3 + \frac{2}{c} x_1 x_3 \leq 1 \\ &&& x > 0 \\ &&& x \in \mathbb{R}^3 \end{aligned}$$

## Geometric Programming: Convex Form

---

GPs are not convex. However, they can easily be converted to a convex form. Consider the posynomial

$$f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

Apply change of variables  $y_i = \log x_i$  (since  $x_i > 0$ ),

$$f(y) = \sum_{k=1}^K \exp\left(\sum_{i=1}^n a_{ik} y_i + b_k\right) = \sum_{k=1}^K \exp(a_k^\top y + b_k)$$

where  $a_k = (a_{1k}, \dots, a_{nk})$  and  $b_k = \log c_k$ .

Taking a logarithm,

$$\log f(y) = \log \left( \sum_{k=1}^K \exp(a_k^\top y + b_k) \right)$$

This is a convex function.

**Definition.** A **convex form GP** is an optimization program of the form

$$\begin{aligned} \text{minimize} \quad & \log \left( \sum_{k=1}^{K_0} \exp(a_k^\top y + b_k) \right) \\ \text{subject to} \quad & c_i^\top y + d_i = 0, \quad \forall 1 \leq i \leq m \\ & \log \left( \sum_{k=1}^{K_j} \exp(e_{kj}^\top y + f_{kj}) \right) \leq 0, \quad \forall 1 \leq j \leq r \\ & y \in \mathbb{R}^n \end{aligned}$$

Note that if  $K_j = 1$  for all  $0 \leq j \leq r$ , this reduces to an LP.

## Eliminating Equality Constraints

---

$$\begin{aligned} f: \mathbb{R}^n &\rightarrow \mathbb{R} & \text{minimize} \quad & f(x) \\ h: \mathbb{R}^n &\rightarrow \mathbb{R}^m & \text{subject to} \quad & h(x) = 0 \\ g: \mathbb{R}^n &\rightarrow \mathbb{R}^r & & g(x) \leq 0 \\ & & & x \in \mathbb{R}^n \end{aligned}$$

Suppose  $\phi: \mathbb{R}^k \rightarrow \mathbb{R}^n$  is such that  $h(x) = 0$  if and only if  $x = \phi(z)$ , for some  $z \in \mathbb{R}^k$ . Then, an equivalent problem is:

$$\begin{aligned} \tilde{f}(z) = f(\phi(z)) & & \text{minimize} \quad & \tilde{f}(z) \\ \tilde{g}(z) = g(\phi(z)) & & \text{subject to} \quad & \tilde{g}(z) \leq 0 \\ & & & z \in \mathbb{R}^k \end{aligned}$$

## Eliminating Equality Constraints

$$\begin{array}{ll} f: \mathbb{R}^n \rightarrow \mathbb{R} & \text{minimize } f(x) \\ A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m & \text{subject to } Ax = b \\ g: \mathbb{R}^n \rightarrow \mathbb{R}^r & g(x) \leq 0 \\ & x \in \mathbb{R}^n \end{array}$$

If  $Ax = b$  has a solution, then all solutions are of the form  $x = Fz + x_0$  for  $z \in \mathbb{R}^k$ , where  $F \in \mathbb{R}^{n \times k}$ , and  $x_0$  is any solution. We can pick  $k = n - \text{rank}(A)$ . Then,

$$\begin{array}{ll} \text{minimize } & f(Fz + x_0) \\ \text{subject to } & g(Fz + x_0) \leq 0 \\ & z \in \mathbb{R}^k \end{array}$$

Note that this problem is of smaller dimension and has no equality constraints. Convexity is also preserved.

## Adding Equality Constraints

Adding equality constraints can be helpful for decomposing and optimization problem into independent subproblems. Consider:

$$\begin{array}{ll} A_i \in \mathbb{R}^{m_i \times n}, b \in \mathbb{R}^{m_i} & \text{minimize } f(A_0x + b_0) \\ f: \mathbb{R}^{m_0} \rightarrow \mathbb{R} & \text{subject to } g_i(A_ix + b_i) \leq 0, \forall 1 \leq i \leq r \\ g_i: \mathbb{R}^{m_i} \rightarrow \mathbb{R} & x \in \mathbb{R}^n \end{array}$$

This is equivalent to:

$$\begin{array}{ll} \text{minimize } & f(y_0) \\ \text{subject to } & g_i(y_i) \leq 0, \quad \forall 1 \leq i \leq r \\ & y_i = A_ix + b, \quad \forall 0 \leq i \leq r \\ & x \in \mathbb{R}^n \\ & y_i \in \mathbb{R}^{m_i}, \quad \forall 0 \leq i \leq r \end{array}$$

This problem has an objective and inequality constraints which are independent.

## Partitioning

---

It is always true that

$$\inf_{x,y} f(x, y) = \inf_x \tilde{f}(x), \quad \text{where } \tilde{f}(x) = \inf_y f(x, y)$$

For example:

$$\begin{array}{ll} f : \mathbb{R}^{n_1} \rightarrow \mathbb{R} & \text{minimize } f(x_1) + g(x_1, x_2) \\ g : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R} & \text{subject to } A_1 x_1 + A_2 x_2 = b \\ A_i \in \mathbb{R}^{m \times n_i}, b \in \mathbb{R}^m & x_i \in \Omega_i, i = 1, 2 \\ \Omega_i \subset \mathbb{R}^{n_i} & \end{array}$$

Equivalent problem:

$$\begin{array}{ll} \tilde{g}(x_1) = \inf_{\substack{A_2 x_2 = c - A_1 x_1, \\ x_2 \in \Omega_2}} g(x_1, x_2) & \text{minimize } f_1(x_1) + \tilde{g}_2(x_1) \\ & \text{subject to } x_1 \in \Omega_1 \end{array}$$

This is useful when  $\tilde{g}(\cdot)$  is easy to compute.

## Epigraph Formulation

---

$$\begin{array}{ll} f : \Omega \rightarrow \mathbb{R} & \text{minimize } f(x) \\ h : \Omega \rightarrow \mathbb{R}^m & \text{subject to } h(x) = 0 \\ g : \Omega \rightarrow \mathbb{R}^r & g(x) \leq 0 \\ \Omega \subset \mathbb{R}^n & x \in \Omega \end{array}$$

Equivalent problem:

$$\begin{array}{ll} \text{minimize } & t \\ \text{subject to } & f(x) - t \leq 0 \\ & h(x) = 0 \\ & g(x) \leq 0 \\ & x \in \Omega \\ & t \in \mathbb{R} \end{array}$$

This is useful when  $f(\cdot)$  is a ‘worst-case’ objective.

## Example: Min-Max Facility Location

---

$$y_i \in \mathbb{R}^n \quad \begin{array}{ll} \text{minimize} & \max_{1 \leq i \leq r} \|x - y_i\| \\ \text{subject to} & x \in \mathbb{R}^n \end{array}$$

Equivalent problem:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \|x - y_i\| \leq t, \quad \forall 1 \leq i \leq n \\ & x \in \mathbb{R}^n \\ & t \in \mathbb{R} \end{array}$$

This is a SOCP.

## Robust Optimization

---

The parameters in an optimization problem are often uncertain. For example, consider the LP:

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & a_i^\top x \leq b_i, \quad \forall 1 \leq i \leq r \\ & x \in \mathbb{R}^n \end{array}$$

Suppose there is some uncertainty in  $\{a_i\}$ . We would like “robust” solutions which do not require knowledge of the precise value of  $a_i$ .

One approach is to require the constraints to hold for all  $a_i$  in a set  $\mathcal{E}_i \subset \mathbb{R}^n$ :

$$\begin{aligned} & \text{minimize} && c^\top x \\ & \text{subject to} && a_i^\top x \leq b_i, \quad \forall a_i \in \mathcal{E}_i, \quad 1 \leq i \leq r \\ & && x \in \mathbb{R}^n \end{aligned}$$

Equivalent problem with worst-case constraints:

$$\begin{aligned} g_i(x) = \sup_{a_i \in \mathcal{E}_i} a_i^\top x & & \text{minimize} && c^\top x \\ & & \text{subject to} && g_i(x) \leq b_i, \quad \forall 1 \leq i \leq r \\ & & && x \in \mathbb{R}^n \end{aligned}$$

Suppose we take  $\mathcal{E}_i$  to be an ellipsoid:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u : \|u\| \leq 1\}$$

Here,  $\bar{a}_i \in \mathbb{R}^n$ ,  $P_i \in \mathbb{R}^{n \times n}$ . The axes of the ellipse are determined by the eigenvalues/eigenvectors of  $P_i^\top P_i$ .

Then,

$$g_i(x) = \sup_{\|u\| \leq 1} (\bar{a}_i + P_i u)^\top x = \bar{a}_i^\top x + \|P_i^\top x\|$$

Thus, the robust LP is a SOCP:

$$\begin{aligned} & \text{minimize} && c^\top x \\ & \text{subject to} && \bar{a}_i^\top x + \|P_i^\top x\| \leq b_i, \quad \forall 1 \leq i \leq r \\ & && x \in \mathbb{R}^n \end{aligned}$$

Suppose we take  $\mathcal{E}_i$  to be a bounded polyhedron,

$$\mathcal{E}_i = \{x \in \mathbb{R}^n : E_i x \leq f_i\}$$

Here,  $E_i \in \mathbb{R}^{m_i \times n}$ ,  $f_i \in \mathbb{R}^{m_i}$ .

Then,

$$g_i(x) = \sup_{E_i y \leq f_i} x^\top y$$

Since  $\mathcal{E}_i$  is bounded, the optimum must occur at a vertex

$$\{\bar{a}_{i,1}, \dots, \bar{a}_{i,k_i}\}.$$

Thus, the robust LP is also an LP:

$$\begin{aligned} & \text{minimize} && c^\top x \\ & \text{subject to} && \bar{a}_{i,j}^\top x \leq b_i, \quad \forall 1 \leq i \leq r, 1 \leq j \leq k_i \\ & && x \in \mathbb{R}^n \end{aligned}$$

## Robust Optimization: Deterministic Model

---

Alternatively, by duality of LPs,

$$g_i(x) = \sup_{E_i y \leq f_i} x^\top y = \inf_{E_i^\top z_i = x, z_i \geq 0} f_i^\top z_i$$

Thus, the robust LP can be expressed as

$$\begin{aligned} & \text{minimize} && c^\top x \\ & \text{subject to} && f_i^\top z_i \leq b_i, \quad \forall 1 \leq i \leq r \\ & && E_i^\top z_i - x = 0, \quad \forall 1 \leq i \leq r \\ & && z_i \geq 0, \quad \forall 1 \leq i \leq r \\ & && z_i \in \mathbb{R}^{m_i} \\ & && x \in \mathbb{R}^n \end{aligned}$$

# Parametric Estimation

---

Suppose we wish to estimate the density  $p(y)$  of a random variable given an observation.

In a **parametric** estimation problem, we would like to find the best guess of  $p(y)$  from a family of densities  $p_x(y)$  indexed by  $x \in \Omega$ .

The **maximum likelihood** (ML) estimation problem is given an observation  $y$  is:

$$\underset{x \in \Omega}{\text{maximize}} \quad p_x(y) \quad \Leftrightarrow \quad \underset{x \in \Omega}{\text{maximize}} \quad \log p_x(y)$$

If  $\Omega \subset \mathbb{R}^n$  is a convex set and the log-likelihood

$$\ell(x) = \log p_x(y)$$

is concave, this is a convex optimization problem.

## Example: Linear Measurements

---

$$y_i = a_i^\top x + v_i, \quad i = 1, \dots, m$$

- $x \in \Omega \subset \mathbb{R}^n$  is a vector of unknown parameters
- $v_i$  is IID measurement noise, with density  $p(z)$
- $y_i$  is a measurement,  $y \in \mathbb{R}^m$  has density

$$p_x(y) = \prod_{i=1}^m p(y_i - a_i^\top x)$$

The ML estimation problem is

$$\begin{aligned} &\underset{x \in \Omega}{\text{maximize}} \quad \ell(x) = \sum_{i=1}^m \log p(y_i - a_i^\top x) \\ &\text{subject to} \quad x \in \Omega \end{aligned}$$

## Example: Linear Measurements

---

**Example.** Gaussian noise:  $p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/2}$ ,  $\sigma > 0$

$$\ell(x) = -\frac{m}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^m (a_i^\top x - y_i)^2$$

The ML estimate is a least-squares regression.

**Example.** Laplacian noise:  $p(z) = \frac{1}{2\alpha} e^{-|z|/\alpha}$ ,  $\alpha > 0$

$$\ell(x) = -m \log 2\alpha - \frac{1}{\alpha} \sum_{i=1}^m |a_i^\top x - y_i|$$

The ML estimate is an  $\ell_1$ -regression.

## Example: Logistic Regression

---

Consider a random variable  $y \in \{0, 1\}$  with distribution

$$p = \mathbf{P}(y = 1) = \frac{\exp(a^\top x)}{1 + \exp(a^\top x)}$$

- $x \in \mathbb{R}^n$  are unknown parameters;  $a \in \mathbb{R}^n$  are observable explanatory variables
- estimation problem: estimate  $x$  from  $m$  observations  $(a_i, y_i)$

Assume that  $y_1 = \dots = y_k = 1$ ,  $y_{k+1} = \dots = y_m = 0$ .

$$\begin{aligned} \ell(x) &= \log \left( \prod_{i=1}^k \frac{\exp(a_i^\top x)}{1 + \exp(a_i^\top x)} \prod_{i=k+1}^m \frac{1}{1 + \exp(a_i^\top x)} \right) \\ &= \sum_{i=1}^k a_i^\top x - \sum_{i=1}^m \log(1 + \exp(a_i^\top x)) \end{aligned}$$

This is concave.

B9824 Foundations of Optimization

## Lecture 9: Optimal Control I

Fall 2008

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### Outline

---

1. Deterministic discrete-time dynamic programming
2. Deterministic continuous-time dynamic programming

Basic optimization problem:

$$\begin{array}{ll} f : \mathcal{C} \rightarrow \mathbb{R} & \text{minimize } f(u) \\ & \text{subject to } u \in \mathcal{C} \end{array}$$

In many cases, the decision we wish to make **decomposes** over “time”, that is

$$u = (u_1, u_2, \dots)$$

If the objective function and constraints also decompose over time, we can seek to exploit this structure. This theory goes under the names sequential decision making, dynamic programming, optimal control, etc.

## Deterministic Discrete-Time Dynamic Program

---

- $k \in \{0, 1, \dots, N\}$ :  
**discrete-time, finite horizon** of length  $N$
- $x_k \in \mathcal{X}_k, k = 0, 1, \dots, N$ :  
**state** of the system at time  $k$   
 $(x_0, x_1, \dots, x_N) =$  state trajectory
- $u_k, k = 0, 1, \dots, N - 1$ :  
**control**, the decision variable to be selected at time  $k$   
 $(u_0, u_1, \dots, u_{N-1}) =$  control trajectory

# Deterministic Discrete-Time Dynamic Program

---

- $u_k \in \mathcal{U}_k(x_k)$ : control **constraint set**, may be state-dependent
- **system dynamics**:  
govern the evolution of the system state, as a **deterministic** function of the controls

$x_0 =$  initial state

$$x_{k+1} = f_k(x_k, u_k) \quad 0 \leq k \leq N - 1 \quad \text{state evolution}$$

- **cost**:  
the objective we seek to minimize

$$\text{minimize } \sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

## Example: Inventory Control

---

A retailer seeks to schedule orders to meet demand, while minimizing costs.

- $x_k =$  inventory level at time  $k$   
 $x_k > 0 \Rightarrow$  excess inventory  
 $x_k < 0 \Rightarrow$  backlogged demand
- $u_k =$  quantity ordered at time  $k$
- $u_k \geq 0 \Rightarrow$  can only order non-negative quantities  
 $x_k + u_k \leq B \Rightarrow$  limited storage capacity
- $d_k =$  demand during time period  $k$

## Example: Inventory Control

---

- system dynamics:

$$x_{k+1} = x_k + u_k - d_k$$

- cost structure:

$r(x_k)$  = penalty for inventory level  $x_k$ ,

e.g., holding cost ( $x_k > 0$ ), or shortage cost ( $x_k < 0$ )

$c(u_k)$  = purchasing cost for  $u_k$  units

$R(x_N)$  = terminal cost for final inventory of  $x_N$

$$\text{minimize } \sum_{k=0}^{N-1} (r(x_k) + c(u_k)) + R(x_N)$$

## Key Structural Properties

---

$$x_{k+1} = f_k(x_k, u_k)$$
$$u_k \in \mathcal{U}_k(x_k)$$

$$\text{minimize } \sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

- underlying **discrete-time dynamical system**  
**Markovian** structure of state evolution
- **additively separable** cost structure

## Variations

---

$$\begin{aligned}x_{k+1} &= f_k(x_k, u_k) \\ u_k &\in \mathcal{U}_k(x_k)\end{aligned}$$

$$\text{minimize } \sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

- Continuous-time
- Infinite horizon: discounted-cost

$$\text{minimize } \sum_{k=0}^{\infty} \alpha^k g_k(x_k, u_k)$$

- Infinite horizon: long-run average-cost

$$\text{minimize } \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

## Variations

---

$$\begin{aligned}x_{k+1} &= f_k(x_k, u_k) \\ u_k &\in \mathcal{U}_k(x_k)\end{aligned}$$

$$\text{minimize } \sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

- Finite/countable/uncountable state/action spaces
- Stochastic dynamic program

$$\begin{aligned}x_{k+1} &= f_k(x_k, u_k, w_k) \\ u_k &\in \mathcal{U}_k(x_k)\end{aligned} \quad \text{minimize } \mathbb{E} \left[ \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) + g_N(x_N) \right]$$

$\{w_k\}$  random parameters (**disturbances**)

- Supply chain management
- Revenue management
- Queueing systems
- Machine learning
- Multi-period portfolio optimization
- Derivatives pricing (e.g., American options)

## DP Principle of Optimality

---

$$\begin{aligned} x_{k+1} &= f_k(x_k, u_k) \\ u_k &\in \mathcal{U}_k(x_k) \end{aligned} \quad \text{minimize} \quad \sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

Suppose that  $\{u_k^*\}$  is an optimal control trajectory, with corresponding state trajectory  $\{x_k^*\}$ . Consider a subproblem starting at time  $1 \leq i \leq N$  with  $x_i = x_i^*$ :

$$\text{minimize} \quad \sum_{k=i}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

Then, the truncated control trajectory  $\{u_k^* : i \leq k < N\}$  must be an optimal solution of the subproblem.

$$\begin{aligned} x_{k+1} &= f_k(x_k, u_k) \\ u_k &\in \mathcal{U}_k(x_k) \end{aligned} \quad \text{minimize} \quad \sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$$

**Definition.** The **optimal cost-to-go function**  $J_k^*(x_k)$  is the optimal cost of the subproblem starting at time  $k$ , with initial state  $x_k$ :

$$J_k^*(x_k) = \text{minimize}_{\substack{u_\ell \in \mathcal{U}(x_\ell) \\ k \leq \ell < N}} \sum_{\ell=k}^{N-1} g_\ell(x_\ell, u_\ell) + g_N(x_N)$$

We define  $J_N^*(x_N) = g_N(x_N)$ .

**Note:**  $J_0^*(x_0)$  is the optimal cost for the original problem, with initial state  $x_0$ .

This is called the **value function** if we are maximizing a reward rather than minimizing a cost.

## DP Algorithm

**Theorem.** Suppose the functions  $\{J_0, J_1, \dots, J_N\}$  are computed according to

$$\begin{aligned} J_N(x_N) &= g_N(x_N), \quad \forall x_N \\ J_k(x_k) &= \min_{u \in \mathcal{U}_k(x_k)} \{g_k(x_k, u) + J_{k+1}(f_k(x_k, u))\}, \quad \forall x_k, 0 \leq k < N \end{aligned}$$

Then,  $J_k$  is equal to the optimal cost-to-go function  $J_k^*$ . Further, suppose the **policy**  $\mu_k^*(x_k)$  satisfies

$$\mu_k^*(x_k) \in \operatorname{argmin}_{u \in \mathcal{U}_k(x_k)} \{g_k(x_k, u) + J_{k+1}(f_k(x_k, u))\}$$

Define state and control trajectories by

$$x_0^* = x_0, \quad u_k^* = \mu_k^*(x_k^*), \quad x_{k+1}^* = f_k(x_k^*, u_k^*)$$

Then, these trajectories are optimal for the original problem.

**Note:** Two-pass algorithm, backward pass to compute cost-to-go functions, forward pass to compute control trajectory.

By induction. Clearly  $J_N = J_N^*$ . Consider  $k < N$ :

$$\begin{aligned} J_k(x_k) &= \min_{u_k \in \mathcal{U}(x_k)} g(x_k, u_k) + J_{k+1}(f_k(x_k, u_k)) \\ &= \min_{u_k \in \mathcal{U}(x_k)} g(x_k, u_k) + J_{k+1}^*(f_k(x_k, u_k)) \\ &= \min_{u_k \in \mathcal{U}(x_k)} g(x_k, u_k) + \min_{\substack{u_\ell \in \mathcal{U}(x_\ell) \\ k+1 \leq \ell < N}} \sum_{\ell=k+1}^{N-1} g_\ell(x_\ell, u_\ell) + g_N(x_N) \\ &= \min_{\substack{u_\ell \in \mathcal{U}(x_\ell) \\ k \leq \ell < N}} \sum_{\ell=k}^{N-1} g_\ell(x_\ell, u_\ell) + g_N(x_N) \\ &= J_k^*(x_k) \end{aligned}$$

### Example: Optimal Consumption

---

A consumer wants to decide between consuming wealth and saving it.

- $x_k$  = total wealth at time  $k$
- $u_k$  = consumption at time  $k$
- $0 \leq u_k \leq x_k$ : can only consume available wealth
- system dynamics:  $r > 0$  return on savings

$$x_{k+1} = (1 + r)(x_k - u_k)$$

- reward structure: maximize cumulative utility of consumption

$$\text{maximize } \sum_{k=0}^{N-1} \sqrt{u_k}$$

## Example: Optimal Consumption

---

$V_k^*(x)$  = optimal value function at time  $k$  with wealth  $x$

$$k = N: \quad V_N^*(x) = 0$$

$$k = N - 1: \quad V_{N-1}^*(x) = \max_{u \in [0, x]} \sqrt{u} = \sqrt{x}$$

$$\mu_{N-1}^*(x) = x$$

$$k = N - 2:$$

$$\begin{aligned} V_{N-2}^*(x) &= \max_{u \in [0, x]} \sqrt{u} + \sqrt{(1+r)(x-u)} \\ &= \sqrt{(1+(1+r))x} \end{aligned}$$

$$\mu_{N-2}^*(x) = x/(1+(1+r))$$

## Example: Optimal Consumption

---

By induction, for  $k < N$ :

$$\begin{aligned} V_k^*(x) &= \max_{u \in [0, x]} \sqrt{u} + \sqrt{(1+r)(x-u) \sum_{\ell=0}^{N-k} (1+r)^\ell} \\ &= \sqrt{x \sum_{\ell=0}^{N-k+1} (1+r)^\ell} \end{aligned}$$

$$\mu_k^*(x) = \frac{x}{\sum_{\ell=0}^{N-k+1} (1+r)^\ell}$$

Suppose that state/control spaces are finite, with  $|\mathcal{X}_k| = X$ ,  
 $|\mathcal{U}_k| = U$ .

What is the complexity, in terms of individual cost function evaluations, of:

- exhaustive search:  $NU^N$
- dynamic programming:  $NXU$

## Practical Considerations

---

- Must be able to solve the single-stage optimization problem *for every*  $x_k$
- Must be able to store the optimal cost-to-go  $J_k^*(x_k)$  *for every*  $x_k$
- This becomes intractable if  $\mathcal{X}_k$  is of high dimension (curse of dimensionality), or is infinite
- Can get lucky when the optimization problem has an analytic solution, and  $J_k^*(x_k)$  takes a simple form
- Approximate dynamic programming!

## Deterministic Continuous-Time Dynamic Program

---

- $t \in [0, T]$

**continuous-time**, **finite horizon** of length  $T$

- $x(t) \in \mathbb{R}^n, t \in [0, T]$

**state** of the system at time  $t$

$\{x(t) : t \in [0, T]\} =$  state trajectory

- $u(t) \in \mathbb{R}^m, t \in [0, T]$

**control**, the decision variable to be selected at time  $t$

$\{u(t) : t \in [0, T]\} =$  control trajectory

We assume admissible control trajectories are **piecewise continuous**

## Deterministic Continuous-Time Dynamic Program

---

- $u(t) \in \mathcal{U}(t) \subset \mathbb{R}^m$ : control **constraint set**

- **system dynamics**:

govern the evolution of the system state, as an ODE

$$x(0) = \text{initial state}$$

$$\frac{d}{dt}x_i(t) = \dot{x}_i(t) = f_i(x(t), u(t), t) \quad \text{state evolution}$$

$$\dot{x}(t) = f(x(t), u(t), t)$$

[We assume existence and uniqueness of a solution for every feasible control trajectory!]

- **cost**:

$$\text{minimize } \int_0^T g(x(t), u(t), t) dt + h(x(T))$$

## Example: Motion Control

---

A unit mass moves along a line under the influence of a limited force. We wish to drive the mass to as close as possible to origin in time  $T$ .

- **state:**  $x(t) = (x_1(t), x_2(t)) = (\text{position}, \text{velocity}) \in \mathbb{R}^2$
- **control:**  $u(t) = (\text{force}) \in \mathbb{R}, |u(t)| \leq 1$
- **system dynamics:**

$$\dot{x}_1(t) = x_2(t) \quad \dot{x}_2(t) = u(t)$$

- **cost:**

$$\text{minimize } \frac{1}{2}|x_1(T)|^2$$

## Example: Calculus of Variations

---

We wish to compute the minimum length of a curve which starts at given point and passes through a given line.

Without loss of generality, we take the point to be  $(0, \alpha)$ , and the line to be the vertical line through  $(T, 0)$ .

Let the curve be defined by  $(t, x(t))$ ,  $t \in [0, T]$ .

$$\begin{aligned} \Rightarrow \quad & \text{minimize } \int_0^T \sqrt{1 + (\dot{x}(t))^2} dt \\ & \text{subject to } x(0) = \alpha \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & \text{minimize } \int_0^T \sqrt{1 + (u(t))^2} dt \\ & \text{subject to } x(0) = \alpha \\ & \dot{x}(t) = u(t) \end{aligned}$$

# Hamilton-Bellman-Jacobi Equation

---

$$\dot{x}(t) = f(x(t), u(t), t) \quad \text{minimize} \quad \int_0^T g(x(t), u(t), t) dt + h(x(T)) \\ u(t) \in \mathcal{U}(t)$$

Heuristic derivation:

$J^*(t, x)$  = optimal cost-to-go function at time  $t$ , state  $x$

DP equations:

$$J^*(T, x) = h(x)$$

$$J^*(t, x) = \min_{\substack{u(\tau) \in \mathcal{U}(\tau) \\ \tau \in [t, T]}} \left[ \int_t^T g(x(\tau), u(\tau), \tau) d\tau + h(x(T)) \mid x(t) = x \right]$$

# Hamilton-Bellman-Jacobi Equation

---

By DP principle of optimality,

$$J^*(t, x) = \min_{\substack{u(\tau) \in \mathcal{U}(\tau) \\ \tau \in [t, t+\delta] \\ x(t)=x}} \int_t^{t+\delta} g(x(\tau), u(\tau), \tau) d\tau + J^*(t + \delta, x(t + \delta)) \\ \approx \min_{\substack{u \in \mathcal{U}(t) \\ x(t)=x}} g(x, u, t)\delta + J^*(t, x) + \frac{\partial}{\partial t} J^*(t, x)\delta \\ \quad + \nabla_x J^*(t, x)^\top \dot{x}(t)\delta \\ = \min_{\substack{u \in \mathcal{U}(t) \\ x(t)=x}} g(x, u, t)\delta + J^*(t, x) + \frac{\partial}{\partial t} J^*(t, x)\delta \\ \quad + \nabla_x J^*(t, x)^\top f(x, u, t)\delta$$

Dividing by  $\delta$  and taking  $\delta \downarrow 0$ ,

$$0 = \min_{u \in \mathcal{U}(t)} \left[ g(x, u, t) + \frac{\partial}{\partial t} J^*(t, x) + \nabla_x J^*(t, x)^\top f(x, u, t) \right]$$

Equivalently,  $\forall (t, x)$ ,

$$\frac{\partial}{\partial t} J^*(t, x) = - \min_{u \in \mathcal{U}(t)} \left[ g(x, u, t) + \nabla_x J^*(t, x)^\top f(x, u, t) \right]$$

Boundary conditions: for all  $x$ ,

$$J^*(T, x) = h(x)$$

This PDE is known as the **Hamilton-Bellman-Jacobi** equation.

## Sufficient Conditions

---

**Theorem.** Suppose that:

- (i)  $J(t, x)$  is a continuously differentiable solution to the HJB equation

$$\frac{\partial}{\partial t} J(t, x) = - \min_{u \in \mathcal{U}(t)} \left[ g(x, u, t) + \nabla_x J(t, x)^\top f(x, u, t) \right]$$

for all  $(t, x)$ , with boundary condition

$$J(T, x) = h(x), \quad \forall x$$

- (ii)  $\mu^*(t, x)$  satisfies, for all  $(t, x)$ ,

$$\mu^*(t, x) \in \operatorname{argmin}_{u \in \mathcal{U}(t)} \left[ g(x, u, t) + \nabla_x J(t, x)^\top f(x, u, t) \right]$$

### Theorem. (con't)

- (iii)  $x^*(t)$  is that state trajectory given control trajectory  $u^*(t) = \mu^*(t, x^*(t))$ . That is,

$$x^*(0) = x(0), \quad \dot{x}^*(t) = f(x^*(t), \mu^*(t, x^*(t)), t)$$

[We assume this solution exists and is unique starting from any  $(t, x)$ , and that  $\mu^*(t, x^*(t))$  is piecewise continuous.]

Then,  $J$  is equal to the optimal cost-to-go function  $J^*$ , and the control trajectory  $u^*(t)$  is optimal.

## Sufficient Conditions: Proof

---

Let  $u(t)$  be a feasible control trajectory and  $x(t)$  the corresponding state trajectory. From the HJB equation,

$$0 \leq g(x(t), u(t), t) + \frac{\partial}{\partial t} J(t, x(t)) + \nabla_x J(t, x(t))^\top f(x, u, t)$$

$$\Rightarrow 0 \leq g(x(t), u(t), t) + \frac{d}{dt} \{J(t, x(t))\}$$

$$\Rightarrow 0 \leq \int_0^T g(x(t), u(t), t) dt + J(T, x(T)) - J(0, x(0))$$

$$\Rightarrow J(0, x(0)) \leq \int_0^T g(x(t), u(t), t) dt + h(x(T))$$

Repeating for  $u^*(t)$  and  $x^*(t)$ ,

$$J(0, x(0)) = \int_0^T g(x^*(t), u^*(t), t) dt + h(x^*(T))$$

Thus,

$$\int_0^T g(x^*(t), u^*(t), t) dt + h(x^*(T)) \leq \int_0^T g(x(t), u(t), t) dt + h(x(T))$$

and  $u^*(t)$  is the optimal control trajectory, and

$$J(0, x(0)) = J^*(0, x(0))$$

We can repeat this argument for any initial  $(t, x)$  to obtain

$$J(t, x) = J^*(t, x)$$

## HJB Equation in Practice

---

HJB equation is non-linear and non-smooth. It can be challenging to solve in practice!

- Guess a form for the solution of the HJB equation, solve for the parameters (analytical or numerical solution)
- Guess a class of ‘good’ control policies, use their cost-to-go functions as a guess to the optimal cost-to-go function, solve for parameters (analytical or numerical solution)
- Numerically solve PDE
- Must be able to solve the minimization efficiently! (e.g., convex optimization problem)

## Example: Motion Control

---

$$\begin{aligned} x(t) &\in \mathbb{R} \\ \dot{x}(t) &= u(t) \\ |u(t)| &\leq 1 \end{aligned} \quad \text{minimize } \frac{1}{2}|x(T)|^2$$

Candidate control policy:

$$\mu(t, x) = -\text{sgn}(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x > 0 \end{cases}$$

Cost-to-go function for  $\mu$ :

$$J_\mu(t, x) = \frac{1}{2} (\max\{0, |x| - (T - t)\})^2$$

## Example: Motion Control

---

HJB calculation:

$$\frac{\partial}{\partial t} J_\mu(t, x) = \max\{0, |x| - (T - t)\}$$

$$\frac{\partial}{\partial x} J_\mu(t, x) = \text{sgn}(x) \max\{0, |x| - (T - t)\}$$

$$0 = \min_{|u| \leq 1} [1 + \text{sgn}(x)u] \max\{0, |x| - (T - t)\}$$

Boundary condition:  $J_\mu(T, x) = \frac{1}{2}x^2$

Optimal solution:

$$J^*(t, x) = \frac{1}{2} (\max\{0, |x| - (T - t)\})^2$$

$$\mu^*(t, x) = \begin{cases} -\text{sgn}(x) & \text{if } |x| > T - t \\ \text{arbitrary} & \text{if } |x| \leq T - t \end{cases}$$

## Example: Linear-Quadratic Control

---

$$x(t) \in \mathbb{R}^n$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$$

$$\text{minimize } \int_0^T \left( x(t)^\top Qx(t) + u(t)^\top Ru(t) \right) dt + x(T)^\top Px(T)$$

$$P, Q \in \mathbb{R}^{n \times n} \text{ symmetric, positive semidefinite}$$

$$R \in \mathbb{R}^{m \times m} \text{ symmetric, positive definite}$$

Guess that the cost-to-go function is quadratic:

$$J(t, x) = x^\top K(t)x$$

Solve for the parameters  $K(t) \in \mathbb{R}^{n \times n}$ , symmetric,  $t \in [0, T]$ .

## Example: Linear-Quadratic Control

---

HJB calculation:

$$\frac{\partial}{\partial t} J(t, x) = x^\top \dot{K}(t)x$$

$$\nabla_x J(t, x) = 2K(t)x$$

$$0 = \min_u \left[ x^\top Qx + u^\top Ru + x^\top \dot{K}(t)x + 2x^\top K(t)(Ax + Bu) \right]$$

Unique optimizer:  $u = -R^{-1}B^\top K(t)x$

$$\Rightarrow 0 = x^\top \left( \dot{K}(t) + K(t)A + A^\top K(t) - K(t)BR^{-1}B^\top K(t) + Q \right) x$$

$$\Rightarrow \dot{K}(t) = K(t)A - A^\top K(t) + K(t)BR^{-1}B^\top K(t) - Q$$

## Example: Linear-Quadratic Control

---

Optimal cost-to-go function/policy:

$$J^*(t, x) = x^\top K(t)x \quad \mu^*(t, x) = -R^{-1}B^\top K(t)x$$

where  $K(t)$  solves the **continuous-time Riccati equation**

$$\dot{K}(t) = K(t)A - A^\top K(t) + K(t)BR^{-1}B^\top K(t) - Q$$

with terminal condition  $K(T) = P$ .

This is a PDE, which is much easier to solve than the original ODE.

B9824 Foundations of Optimization

## Lecture 10: Optimal Control II

Fall 2008

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### Outline

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1. Pontryagin Minimum Principle
2. Applications & extensions

## HJB Equation

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t)) dt + h(x(T))$$

[Note: no time dependence for  $f$ ,  $g$ , and  $\mathcal{U}$ . We will revisit this later.]

HJB equation:

$$0 = \min_{u \in \mathcal{U}} \left[ g(x, u) + \frac{\partial}{\partial t} J^*(t, x) + \nabla_x J^*(t, x)^\top f(x, u) \right], \quad \forall (t, x)$$
$$J^*(T, x) = h(x), \quad \forall x$$

If a solution exists, the optimal policy takes the form:

$$\mu^*(t, x) \in \operatorname{argmin}_{u \in \mathcal{U}} \left[ g(x, u) + \nabla_x J^*(t, x)^\top f(x, u) \right]$$

In order to determine the optimal policy, we need to solve for  $J^*(t, x)$  for **all** states  $(t, x)$ .

## HJB Equation

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t)) dt + h(x(T))$$

Optimal state trajectory:

$$x^*(0) = x(0), \quad \dot{x}^*(t) = f(x^*(t), u^*(t))$$

Optimal control trajectory:

$$u^*(t) = \operatorname{argmin}_{u \in \mathcal{U}} \left[ g(x^*(t), u) + \nabla_x J^*(t, x^*(t))^\top f(x^*(t), u) \right]$$

In order to determine the optimal control trajectory, we only need to know  $\nabla_x J(t, x)$  **along the optimal state trajectory**  $(t, x^*(t))$ , that is, the function

$$p(t) \triangleq \nabla_x J^*(t, x^*(t)), \quad \forall t \in [0, T]$$

# Pontryagin Minimum Principle

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t)) dt + h(x(T))$$

$$\begin{aligned} \dot{x}^*(t) &= f(x^*(t), u^*(t)) \\ u^*(t) &= \operatorname{argmin}_{u \in \mathcal{U}} \left[ g(x^*(t), u) + p(t)^\top f(x^*(t), u) \right] \\ p(t) &\triangleq \nabla_x J^*(t, x^*(t)) \end{aligned}$$

The **Pontryagin Minimum Principle** provides an **adjoint equation** that allows the computation of  $p(t)$  without knowledge of  $J^*(t, x)$ .

- It is often much easier to compute  $p(t)$  than  $J^*(t, x)$
- It applies much more generally, even in cases where  $J^*(t, x)$  is not differentiable and the HJB equation has no solution

## A Sensitivity Lemma

---

$$F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$$

**Lemma.** Suppose  $F(y, u)$  is continuously differentiable, and let  $U \subset \mathbb{R}^m$  be a convex set. Assume that  $\mu^*(y)$  is a continuously differentiable function such that

$$\mu^*(y) = \operatorname{argmin}_{u \in U} F(y, u), \quad \forall y$$

Then,

$$\nabla_y \left\{ \min_{u \in U} F(y, u) \right\} = \nabla_y F(y, \mu^*(y))$$

## Sensitivity Lemma: Proof

---

Note that

$$\min_{u \in U} F(y, u) = F(y, \mu^*(y))$$

By the chain rule,

$$\begin{aligned}\nabla_y \left\{ \min_{u \in U} F(y, u) \right\} &= \nabla_y \{ F(y, \mu^*(y)) \} \\ &= \nabla_y F(y, \mu^*(y)) + \nabla \mu^*(y) \nabla_u F(y, \mu^*(y))\end{aligned}$$

It suffices to prove that

$$\nabla \mu^*(y) \nabla_u F(y, \mu^*(y)) = 0$$

Since  $F(y, \cdot)$  is continuously differentiable and  $U$  convex,

$$(u - \mu^*(y))^\top \nabla_u F(y, \mu^*(y)) \geq 0, \quad \forall u \in U$$

## Sensitivity Lemma: Proof

---

For some  $d \in \mathbb{R}^m$ , and  $\alpha > 0$ , set  $u = \mu^*(y + \alpha d)$ . By the mean value theorem,

$$u = \mu^*(y) + \alpha \nabla \mu^*(\tilde{y})^\top d$$

where  $\tilde{y}$  is on the line segment between  $y$  and  $y + \alpha d$ .

Then,

$$\alpha d^\top \nabla \mu^*(\tilde{y}) \nabla_u F(y, \mu^*(y)) \geq 0$$

Dividing by  $\alpha$  and taking  $\alpha \downarrow 0$ ,

$$d^\top \nabla \mu^*(y) \nabla_u F(y, \mu^*(y)) \geq 0$$

Since this is true for all  $d$ , the result follows.

# Pontryagin Minimum Principle

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t)) dt + h(x(T))$$

We will informally derive the Pontryagin Minimum Principle, under the assumption that

- $\mathcal{U}$  is convex
- A solution  $J^*(t, x)$  to the HJB equation exists, and the corresponding  $\mu^*(t, x)$  is continuously differentiable

HJB equation:

$$0 = \min_{u \in \mathcal{U}} \left[ g(x, u) + \nabla_t J^*(t, x) + \nabla_x J^*(t, x)^\top f(x, u) \right], \quad \forall (t, x)$$

## Differentiating HJB

---

Differentiating with respect to  $(t, x)$ , and applying the sensitivity lemma:

$$\begin{aligned} 0 &= \nabla_x g(x, \mu^*(t, x)) + \nabla_{xt}^2 J^*(t, x) + \nabla_{xx}^2 J^*(t, x) f(x, \mu^*(t, x)) \\ &\quad + \nabla_x f(x, \mu^*(t, x)) \nabla_x J^*(t, x) \end{aligned}$$

$$0 = \nabla_{tt}^2 J^*(t, x) + \nabla_{xt}^2 J^*(t, x)^\top f(x, \mu^*(t, x))$$

Since this holds for every  $(t, x)$ , we can evaluate it along the optimal trajectory  $(t, x^*(t))$ .

$$\dot{x}^*(t) = f(x^*(t), u^*(t)), \quad u^*(t) = \mu^*(t, x^*(t))$$

By the chain rule,

$$\frac{d}{dt} \{ \nabla_x J^*(t, x^*(t)) \} = \nabla_{xt}^2 J^*(t, x^*(t)) + \nabla_{xx}^2 J^*(t, x^*(t)) f(x^*(t), u^*(t))$$

$$\frac{d}{dt} \{ \nabla_t J^*(t, x^*(t)) \} = \nabla_{tt}^2 J^*(t, x^*(t)) + \nabla_{xt}^2 J^*(t, x^*(t))^\top f(x^*(t), u^*(t))$$

Define

$$p(t) \triangleq \nabla_x J^*(t, x^*(t)), \quad p_0(t) \triangleq \nabla_t J^*(t, x^*(t))$$

Then, the differentiated HJB equation becomes

$$\frac{d}{dt} p(t) = -\nabla_x f(x^*(t), u^*(t)) p(t) - \nabla_x g(x^*(t), u^*(t)), \quad \frac{d}{dt} p_0(t) = 0$$

The first equation is known as the **adjoint equation**.

Boundary condition:  $J^*(T, x) = h(x)$

Differentiating with respect to  $x$ ,

$$p(T) = \nabla h(x^*(T))$$

Finally,  $u^*(t)$  must satisfy

$$\begin{aligned} u^*(t) &\in \operatorname{argmin}_{u \in \mathcal{U}} \left[ g(x^*(t), u) + \nabla_x J^*(t, x^*(t))^\top f(x^*(t), u) \right] \\ &= \operatorname{argmin}_{u \in \mathcal{U}} \left[ g(x^*(t), u) + p(t)^\top f(x^*(t), u) \right] \end{aligned}$$

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t)) dt + h(x(T))$$

**Definition.** Define the **Hamiltonian** function  $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by

$$H(x, u, p) \triangleq g(x, u) + p^\top f(x, u)$$

The system, adjoint, and optimal control equations can be compactly written as

$$\begin{aligned} \dot{x}^*(t) &= \nabla_p H(x^*(t), u^*(t), p(t)) \\ \dot{p}(t) &= -\nabla_x H(x^*(t), u^*(t), p(t)) \\ u^*(t) &= \underset{u \in \mathcal{U}}{\operatorname{argmin}} H(x^*(t), u, p(t)) \end{aligned}$$

## Pontryagin Minimum Principle

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t)) dt + h(x(T))$$

Informally, we have derived the following:

**Minimum Principle.** Let  $u^*(t)$  be an optimal control trajectory and  $x^*(t)$  the corresponding state trajectory. Let  $p(t)$  be the solution of the adjoint equation

$$\dot{p}(t) = -\nabla_x H(x^*(t), u^*(t), p(t))$$

with the boundary condition

$$p(T) = \nabla h(x^*(T))$$

Then,

$$u^*(t) = \underset{u \in \mathcal{U}}{\operatorname{argmin}} H(x^*(t), u, p(t)), \quad \forall t \in [0, T]$$

and, for some constant  $C$ ,

$$H(x^*(t), u^*(t), p(t)) = C, \quad \forall t \in [0, T]$$

## Pontryagin Minimum Principle: Remarks

---

Our informal derivation assumes conditions (solution to the HJB equation) that are difficult to verify

The name ‘Minimum Principle’ comes from the fact that the optimal control minimizes the Hamiltonian,

$$H(x^*(t), u^*(t), p(t)) \leq H(x^*(t), u(t), p(t))$$

for all feasible control paths  $u(t)$ .

The variables  $p(t)$  are dual variables to the system dynamics ‘constraint’

$$\dot{x}(t) = f(x(t), u(t))$$

A discrete-time version is also available

## Pontryagin Minimum Principle: Remarks

---

Minimum Principle provides **necessary** but not sufficient conditions for optimality. To guarantee optimality:

- Prove an optimal control trajectory exists
- Find all paths satisfying conditions of the minimum principle, pick lowest cost path
- If  $f$  is linear in  $(x, u)$ ,  $\mathcal{U}$  is convex, and  $g, h$  are convex, Minimum Principle is necessary and sufficient

Suppose we solve

$$u^*(t) = \underset{u \in \mathcal{U}}{\operatorname{argmin}} H(x^*(t), u, p(t))$$

for  $u^*(t)$  as a function of  $x^*(t)$  and  $p(t)$

Then, we have a system of  $2n$  ODEs

$$\dot{x}^*(t) = \nabla_p H(x^*(t), u^*(t), p(t)), \quad \dot{p}(t) = -\nabla_x H(x^*(t), u^*(t), p(t))$$

with  $2n$  'split' boundary conditions

$$x^*(0) = x_0, \quad p(T) = \nabla h(x^*(T))$$

In principle, these can be solved numerically

## Example: Calculus of Variations

---

$$\begin{aligned} \dot{x}(t) &= u(t) \\ u(t) &\in \mathbb{R} \\ x(0) &= \alpha \end{aligned} \quad \text{minimize} \quad \int_0^T \sqrt{1 + u(t)^2} dt$$

Hamiltonian:  $H(x, u, p) = \sqrt{1 + u^2} + pu$

Adjoint equation:  $\dot{p}(t) = 0, p(T) = 0 \Rightarrow p(t) = 0$

Optimal control:  $u^*(t) = \underset{u}{\operatorname{argmin}} \sqrt{1 + u^2} = 0$

Optimal state:  $\dot{x}^*(t) = 0, x^*(0) = \alpha \Rightarrow p(t) = 0$

Here, since the system dynamics are linear and the cost is convex, these conditions are necessary and sufficient.

## Example: Resource Allocation

---

$$\begin{aligned} \dot{x}(t) &= \gamma u(t)x(t) \\ 0 &\leq u(t) \leq 1 \\ x(0) &= x_0 \\ x_0 > 0, \gamma > 0 \end{aligned} \quad \text{maximize} \quad \int_0^T (1 - u(t))x(t) dt$$

Hamiltonian:  $H(x, u, p) = (1 - u)x + p\gamma ux$

Adjoint equation:  $\dot{p}(t) = -\gamma u^*(t)p(t) - 1 + u^*(t), \quad p(T) = 0$

Optimal control:

$$u^*(t) = \underset{0 \leq u \leq 1}{\operatorname{argmax}} H(x^*(t), u, p(t)) \quad \Rightarrow \quad u^*(t) = \begin{cases} 0 & \text{if } p(t) < 1/\gamma \\ 1 & \text{if } p(t) \geq 1/\gamma \end{cases}$$

## Example: Resource Allocation

---

$$\begin{aligned} \dot{x}(t) &= \gamma u(t)x(t) \\ 0 &\leq u(t) \leq 1 \\ x(0) &= x_0 \\ x_0 > 0, \gamma > 0 \end{aligned} \quad \text{maximize} \quad \int_0^T (1 - u(t))x(t) dt$$

Since  $p(T) = 0$ , if  $t$  is close to  $T$ ,  $p(t) < 1/\gamma$ , thus  $u^*(t) = 1$ , and

$$\dot{p}(t) = -1 \quad \Rightarrow \quad p(t) = T - t \text{ if } t \geq \tau \triangleq T - 1/\gamma$$

If  $t < \tau$ ,  $u^*(t) = 1$ , thus

$$\dot{p}(t) = -\gamma p(t) \quad \Rightarrow \quad p(t) = \frac{1}{\gamma} e^{-\gamma(t-\tau)} \text{ if } t < \tau$$

Optimal state:

$$x^*(t) = \begin{cases} x_0 e^{\gamma t} & \text{if } t < \tau \\ x_0 e^{\gamma \tau} & \text{if } t \geq \tau \end{cases}$$

## Example: Linear-Quadratic

---

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) \\ u(t) &\in \mathbb{R} \\ x(0) &= x_0 \\ q &> 0 \end{aligned} \quad \text{minimize } \frac{1}{2} \int_0^T u(t)^2 dt + \frac{1}{2} qx(T)^2$$

Hamiltonian:  $H(x, u, p) = \frac{1}{2}u^2 + p(ax + bu)$

Adjoint equation:  $\dot{p}(t) = -ap(t), p(T) = qx^*(T) \Rightarrow p(t) = \zeta e^{-at}$

Optimal control:

$$u^*(t) = \underset{u}{\operatorname{argmin}} \frac{1}{2}u^2 + p(t)(ax(t) + bu) \Rightarrow u^*(t) = -bp(t)$$

## Example: Linear-Quadratic

---

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) \\ u(t) &\in \mathbb{R} \\ x(0) &= x_0 \\ q &> 0 \end{aligned} \quad \text{minimize } \frac{1}{2} \int_0^T u(t)^2 dt + \frac{1}{2} qx(T)^2$$

Optimal state:

$$\begin{aligned} \dot{x}^*(t) &= ax^*(t) - b^2\zeta e^{-at}, x(0) = x_0 \\ \Rightarrow x^*(t) &= x_0 e^{at} + \frac{b^2\zeta}{2a} (e^{-at} - e^{at}) \end{aligned}$$

$\zeta$  is determined by  $p(T) = qx^*(T)$ :

$$\zeta e^{-aT} = qx_0 e^{aT} + \frac{qb^2\zeta}{2a} (e^{-aT} - e^{aT})$$

## Extension: Fixed Terminal State

---

Suppose that, in addition to the initial state  $x(0) = x_0$ , the terminal state  $x(T) = x_T$  is given. Our informal derivation continues to hold, except that:

$$J^*(T, x) = \begin{cases} 0 & \text{if } x = x(T) \\ \infty & \text{otherwise} \end{cases}$$

This is not differentiable, thus the boundary condition  $p(T) = \nabla h(x^*(T))$  does not hold.

However, as compensation, we have  $n$  additional boundary conditions

$$x^*(T) = x_T$$

## Extension: Fixed Terminal State

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \\ x(T) &= x_T \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t)) dt$$

Minimum Principle:

$$\begin{aligned} \dot{x}^*(t) &= \nabla_p H(x^*(t), u^*(t), p(t)), & \dot{p}(t) &= -\nabla_x H(x^*(t), u^*(t), p(t)) \\ u^*(t) &= \underset{u \in \mathcal{U}}{\operatorname{argmin}} H(x^*(t), u, p(t)) \\ x^*(0) &= x_0, & x^*(T) &= x_T \end{aligned}$$

Generalization: some components of the terminal state are fixed

## Example: Minimum Length Curve Revisited

---

$$\begin{aligned} \dot{x}(t) &= u(t) \\ u(t) &\in \mathbb{R} \\ x(0) &= \alpha \\ x(T) &= \beta \end{aligned} \quad \text{minimize} \quad \int_0^T \sqrt{1 + u(t)^2} dt$$

Hamiltonian:  $H(x, u, p) = \sqrt{1 + u^2} + pu$

Adjoint equation:  $\dot{p}(t) = 0 \Rightarrow p(t) = \text{constant}$

Optimal control:  $u^*(t) = \underset{u}{\operatorname{argmin}} \sqrt{1 + u^2} + p(t)u = \text{constant}$

Optimal state:  $\dot{x}^*(t) = \text{constant}, x^*(0) = \alpha, x^*(T) = \beta$

The optimal trajectory is a straight line

## Extension: Free Initial State

---

Suppose that the initial state is not given, but subject to optimization. Then,

$$J^*(0, x^*(0)) \leq J^*(0, x), \quad \forall x \in \mathbb{R}^n$$

$$\nabla_x J^*(0, x^*(0)) = 0 \quad \Rightarrow \quad p(0) = 0$$

Generalization: If there is a cost  $\ell(x)$  on the initial state,

$$\ell(x^*(0)) + J^*(0, x^*(0)) \leq \ell(x) + J^*(0, x), \quad \forall x \in \mathbb{R}^n$$

$$\nabla \ell(x^*(0)) + \nabla_x J^*(0, x^*(0)) = 0 \quad \Rightarrow \quad p(0) = -\nabla \ell(x^*(0))$$

## Extension: Free Initial State

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in \mathcal{U} \\ x(0) &\text{ free} \end{aligned} \quad \text{minimize } \ell(x(0)) + \int_0^T g(x(t), u(t)) dt + h(x(T))$$

Minimum Principle:

$$\begin{aligned} \dot{x}^*(t) &= \nabla_p H(x^*(t), u^*(t), p(t)), & \dot{p}(t) &= -\nabla_x H(x^*(t), u^*(t), p(t)) \\ u^*(t) &= \underset{u \in \mathcal{U}}{\operatorname{argmin}} H(x^*(t), u, p(t)) \\ p(0) &= -\nabla \ell(x^*(0)), & p(T) &= \nabla h(x^*(T)) \end{aligned}$$

## Extension: Free Terminal Time

---

Suppose the initial and/or state is given, but the terminal time  $T$  is subject to optimization.

Let  $(x^*(t), u^*(t))$  be the optimal trajectory, and  $T^*$  the optimal termination time.  $(x^*(t), u^*(t))$  is still optimal for a fixed time horizon of  $T = T^*$ . Thus,

$$u^*(t) = \underset{u \in \mathcal{U}}{\operatorname{argmin}} H(x^*(t), u, p(t)), \quad \forall t \in [0, T^*]$$

with  $(x^*(t), p(t))$  given by system-adjoint equations.

Further,

$$\nabla_t J^*(0, x^*(0)) = 0$$

From the HJB equation along the optimal path,

$$\begin{aligned}\nabla_t J^*(0, x^*(0)) &= -H(x^*(t), u^*(t), p(t)) \\ \Rightarrow H(x^*(0), u^*(0), p(0)) &= 0\end{aligned}$$

Then,

$$H(x^*(t), u^*(t), p(t)) = 0, \quad \forall t \in [0, T^*]$$

## Example: Minimum Time Problem

---

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \quad \dot{x}_2(t) = u(t) \\ |u(t)| &\leq 1 \\ x_1(0) &= \alpha, \quad x_2(0) = \gamma \\ x_1(T) &= x_2(T) = 0 \\ T &\text{ free}\end{aligned} \quad \text{minimize } T = \int_0^T 1 \, dt$$

Hamiltonian:  $H(x, u, p) = 1 + p_1 x_2 + p_2 u$

Optimal control:

$$\begin{aligned}u^*(t) &= \underset{|u| \leq 1}{\operatorname{argmin}} 1 + p_1(t)x_2^*(t) + p_2(t)u \\ \Rightarrow u^*(t) &= \begin{cases} 1 & \text{if } p_2(t) < 0 \\ -1 & \text{if } p_2(t) \geq 0 \end{cases}\end{aligned}$$

## Example: Minimum Time Problem

---

Adjoint equation:

$$\begin{aligned}\dot{p}_1(t) &= 0, & \dot{p}_2(t) &= -p_1(t) \\ \Rightarrow p_1(t) &= c_1, & p_2(t) &= c_2 - c_1 t\end{aligned}$$

$p_2(t)$  can change sign at most once, call this switching time  $\tau$

State trajectory: for  $t$  before  $\tau$ , suppose  $u(t) = \zeta \in \{\pm 1\}$

$$\begin{aligned}x_2(t) &= x_2(0) + \zeta t, & x_1(t) &= x_1(0) + x_2(0)t + \frac{1}{2}\zeta t^2 \\ \Rightarrow x_1(t) - \frac{1}{2\zeta}x_2(t)^2 &= x_1(0) - \frac{1}{2\zeta}x_2(0)^2\end{aligned}$$

Switching curve:  $x_1 = -\operatorname{sgn}(x_2)\frac{1}{2}x_2^2$

## Extension: Time-Varying System and Cost

---

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), t) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0\end{aligned} \quad \text{minimize } \int_0^T g(x(t), u(t), t) dt + h(x(T))$$

We can introduce a new state variable  $y(t)$  representing time by

$$y(0) = 0, \quad \dot{y}(t) = 1$$

and apply the existing Minimum Principle.

## Extension: Time-Varying System and Cost

---

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ u(t) &\in \mathcal{U} \\ x(0) &= x_0 \end{aligned} \quad \text{minimize} \quad \int_0^T g(x(t), u(t), t) dt + h(x(T))$$

Equivalently, define

$$H(x, u, p, t) \triangleq g(x, u, t) + p^\top f(x, u, t)$$

Then,

$$\begin{aligned} \dot{x}^*(t) &= \nabla_p H(x^*(t), u^*(t), p(t), t) \\ \dot{p}(t) &= -\nabla_x H(x^*(t), u^*(t), p(t), t) \\ u^*(t) &= \underset{u \in \mathcal{U}}{\operatorname{argmin}} H(x^*(t), u, p(t), t) \\ x^*(0) &= x_0, \quad p(T) = \nabla h(x^*(T)) \end{aligned}$$

However,  $H(x^*(t), u^*(t), p(t), t)$  is not constant.

B9824 Foundations of Optimization

## Lecture 11: Vector Space Optimization I

Fall 2008

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### Outline

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1. Vector spaces, Banach spaces
2. Weierstrass' theorem
3. Inner product spaces, Hilbert spaces
4. Projection theorem

**Definition.** A **vector space** is a set  $\mathcal{X}$  equipped with the operations of

- **addition:**  $x, y \in \mathcal{X} \Rightarrow x + y \in \mathcal{X}$
- **scalar multiplication:**  $x \in \mathcal{X}, \alpha \in \mathbb{R} \Rightarrow \alpha x \in \mathcal{X}$

that satisfies the axioms

1.  $x + y = y + x$  (commutative)
2.  $(x + y) + z = x + (y + x)$  (associative)
3. there exists  $\mathbf{0} \in \mathcal{X}$  with  $x + \mathbf{0} = x$
4.  $\alpha(x + y) = \alpha x + \alpha y$  (distributive)
5.  $(\alpha + \beta)x = \alpha x + \beta x$  (distributive)
6.  $(\alpha\beta)x = \alpha(\beta x)$  (associative)
7.  $0x = \mathbf{0}, \quad 1x = x$

## Examples

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**Example.**  $\mathcal{X} = \mathbb{R}^n$

**Example.**  $\mathcal{X} = \mathbb{R}^\infty$ , the set of (countably) infinite sequences of real numbers

**Example.**  $\mathcal{X} = \{x \in \mathbb{R}^\infty : x \text{ has finitely many non-zero terms}\}$

**Example.**  $\mathcal{X} = c_0 \triangleq \{x \in \mathbb{R}^\infty : \lim_{n \rightarrow \infty} x_n = 0\}$

**Example.**  $\mathcal{X} = C[a, b] \triangleq \{\text{continuous functions from } [a, b] \text{ to } \mathbb{R}\}$

**Definition.** The set  $\mathcal{C} \subset \mathcal{X}$  is a **subspace** if, for all points  $x, y \in \mathcal{C}$ , and scalars  $\alpha, \beta \in \mathbb{R}$ ,

$$\alpha x + \beta y \in \mathcal{C}$$

**Definition.** The set  $\mathcal{C} \subset \mathcal{X}$  is **affine** (linear variety) if, for all points  $x, y \in \mathcal{C}$ , and scalars  $\lambda \in \mathbb{R}$ ,

$$\lambda x + (1 - \lambda)y \in \mathcal{C}$$

**Definition.** The set  $\mathcal{C} \subset \mathcal{X}$  is **convex** if, for all points  $x, y \in \mathcal{C}$ , and scalars  $0 \leq \lambda \leq 1$ ,

$$\lambda x + (1 - \lambda)y \in \mathcal{C}$$

**Definition.** The set  $\mathcal{C} \subset \mathcal{X}$  is a **cone** if, for all points  $x \in \mathcal{C}$ , and scalars  $\lambda \geq 0$ ,

$$\lambda x \in \mathcal{C}$$

## Linear Independence, Dimension

---

**Definition.** A **linear combination** of the vectors  $x_1, \dots, x_n$  is a sum of the form

$$\alpha_1 x_1 + \dots + \alpha_n x_n$$

The subspace generated by the set  $\mathcal{S} \subset \mathcal{X}$  is the set of all linear combinations of vectors in  $\mathcal{S}$ .

**Definition.** A vector  $x \in \mathcal{X}$  is **linearly independent** from a set  $\mathcal{S} \subset \mathcal{X}$  if it cannot be expressed as a linear combination of elements of  $\mathcal{S}$ .

**Definition.** A set of vectors  $\mathcal{S} \subset \mathcal{X}$  is **linear independent** if, for each  $x \in \mathcal{S}$ ,  $x$  is linearly independent from  $\mathcal{S} \setminus \{x\}$ .

**Definition.** A linearly independent, finite set  $\mathcal{S} \subset \mathcal{X}$  is said to be a (finite) **basis** if  $\mathcal{S}$  generates  $\mathcal{X}$ . If a vector space has a finite basis, it is called **finite dimensional**, otherwise it is called **infinite dimensional**.

**Definition.** A **normed vector space** is a vector space  $\mathcal{X}$  associated with a real-valued function  $\|\cdot\|$  on  $\mathcal{X}$  such that

1.  $\|x\| = 0$  if and only if  $x = \mathbf{0}$
2.  $\|\alpha x\| = |\alpha|\|x\|$
3.  $\|x + y\| \leq \|x\| + \|y\|$

## Examples

---

**Example.**  $C[a, b]$ ,  $\|x\| \triangleq \max_{a \leq t \leq b} |x(t)|$

**Example.**

$D[a, b] \triangleq \{\text{continuously differentiable functions from } [a, b] \text{ to } \mathbb{R}\}$

$$\|x\| \triangleq \max_{a \leq t \leq b} |x(t)| + \max_{a \leq t \leq b} |\dot{x}(t)|$$

**Example.** finitely non-zero sequences,  $\|x\| \triangleq \sum_{i=1}^{\infty} |x_i|$

**Example.** real-valued continuous functions over  $[a, b]$ ,

$$\|x\| \triangleq \int_a^b |x(t)| dt$$

**Definition.** For  $1 \leq p \leq \infty$ , the normed linear space  $\ell_p$  is the space of all sequences  $x \in \mathbb{R}^\infty$  with

$$\sum_{i=1}^{\infty} |x_i|^p < \infty, \quad \text{if } p < \infty$$
$$\sup_i |x_i| < \infty, \quad \text{if } p = \infty$$

with norm

$$\|x\|_p = \begin{cases} \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} & \text{if } p < \infty \\ \sup_i |x_i| & \text{if } p = \infty \end{cases}$$

**Definition.** For  $1 \leq p < \infty$ , the normed linear space  $L_p[a, b]$  is the space of all measurable, real-valued functions  $x : [a, b] \rightarrow \mathbb{R}$  where  $|x(t)|^p$  is Lebesgue integrable, with norm

$$\|x\|_p = \left( \int_a^b |x(t)|^p dt \right)^{1/p}$$

**Note:** Functions in  $L_p[a, b]$  are considered equal if they differ only on a set of measure zero. For example,  $\|x\|_p = 0$  implies that  $x(t) = 0$  except possibly on a set of measure zero, we identify all such functions with the element  $\mathbf{0} \in L_p[a, b]$ .

**Definition.** The normed linear space  $L_\infty[a, b]$  is the space of all measurable, real-valued functions  $x : [a, b] \rightarrow \mathbb{R}$  that are bounded except possibly on a set of measure zero, with norm

$$\|x\|_\infty = \inf_{y(t)=x(t) \text{ a.e.}} \sup_{a \leq t \leq b} |y(t)|$$

## Basic Topology

---

Consider a normed linear space  $\mathcal{X}$ .

**Definition.** An **open ball** (or, “neighborhood”) around a point  $x \in \mathcal{X}$  with radius  $r > 0$  is the set

$$N_r(x) \triangleq \{y \in \mathcal{X} : \|x - y\| < r\}$$

Consider a set  $\mathcal{E} \subset \mathcal{X}$ .

**Definition.** A point  $x \in \mathcal{E}$  is an **interior point** if there exists an open ball  $N_r(x)$  such that  $N_r(x) \subset \mathcal{E}$ . The **interior**  $\text{int } \mathcal{E}$  is defined to be the set of all interior points of  $\mathcal{E}$ .

**Definition.**  $\mathcal{E}$  is **open** if  $\mathcal{E} = \text{int } \mathcal{E}$ .

**Definition.** A point  $x \in \mathcal{X}$  is a **closure point** of  $\mathcal{E}$  if, for every open ball  $N_r(x)$ , there exists  $y \in \mathcal{E}$  with  $y \in N_r(x)$ . The **closure**  $\text{cl } \mathcal{E}$  is defined to be the set of all closure points of  $\mathcal{E}$ .

**Definition.**  $\mathcal{E}$  is **closed** if every closure point if  $\mathcal{E} = \text{cl } \mathcal{E}$ .

## Convergence

---

**Definition.** A sequence of vectors  $\{x_k\} \subset \mathcal{X}$  **converges** to a limit  $x \in \mathcal{X}$  if

$$\lim_{k \rightarrow \infty} \|x - x_k\| = 0$$

We say  $x_k \rightarrow x$ .

Consider a set  $\mathcal{E} \subset \mathcal{X}$ .

**Definition.**  $\mathcal{E}$  is **compact** if, given a sequence  $\{x_k\} \subset \mathcal{E}$ , there is a subsequence  $\{x_{k_i}\}$  converging to an element  $x \in \mathcal{E}$ .

**Note:** In infinite dimensional vector spaces, compactness is not equivalent to being closed and bounded!

## Transformations

---

**Definition.** If  $\mathcal{X}$  and  $\mathcal{Y}$  are two vector spaces, a **transformation** is a function  $T : \mathcal{X} \rightarrow \mathcal{Y}$ .

**Definition.** If  $\mathcal{X}$  is a vector space, a **functional** is a map  $f : \mathcal{X} \rightarrow \mathbb{R}$ .

**Definition.** A transformation  $T : \mathcal{X} \rightarrow \mathcal{Y}$  between two normed vector spaces is **continuous** at the point  $x \in \mathcal{X}$  if, for every sequence  $\{x_k\} \subset \mathcal{X}$  with  $x_k \rightarrow x$ ,

$$T(x_k) \rightarrow T(x)$$

We say  $T(\cdot)$  is **continuous** if it is continuous at all points of  $\mathcal{X}$ .

**Theorem.** (Weierstrass) Let  $\mathcal{X}$  be a normed linear space, and  $\mathcal{C} \subset \mathcal{X}$  a non-empty, compact set. If  $f : \mathcal{C} \rightarrow \mathbb{R}$  is a continuous function, then the optimization program

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

has a globally optimal solution.

## Weierstrass' Theorem: Proof

---

Since  $\mathcal{C}$  is non-empty,

$$M = \inf_{x \in \mathcal{C}} f(x)$$

(allowing for  $M = -\infty$ ). Then, there exists a sequence  $x_k \in \mathcal{C}$  with  $f(x_k) \rightarrow M$ .

By compactness, there must exist a convergent subsequence  $\{x_{k_i}\}$ , with  $x_{k_i} \rightarrow x \in \mathcal{C}$ . By continuity, we must have  $f(x) = M$ . Thus,  $M < \infty$ , and the global optimum is achieved by  $x$ .

Let  $\mathcal{X}$  be a normed linear space.

**Definition.** A sequence  $\{x_n\} \subset \mathcal{X}$  is a **Cauchy sequence** if, given  $\epsilon > 0$ , there is an integer  $N$  such that

$$\|x_n - x_m\| < \epsilon$$

for all  $m, n > N$ .

**Definition.** A normed linear space  $\mathcal{X}$  is **complete** if every Cauchy sequence converges. Such a space is known as a **Banach space**.

## Examples

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**Example.**  $C[a, b]$  (with sup-norm) is a Banach space

**Example.**  $\ell_p$ ,  $1 \leq p \leq \infty$ , is a Banach space

**Example.**  $L_p[a, b]$ ,  $1 \leq p \leq \infty$ , is a Banach space

**Definition.** An **inner product space**  $\mathcal{X}$  is a vector space together with an inner product  $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  that satisfies the axioms

1.  $\langle x, y \rangle = \langle y, x \rangle$  (symmetry)
2.  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$  (linearity)
3.  $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$  (linearity)
4.  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0$  if and only if  $x = 0$  (positive definiteness)

Given an inner product space  $\mathcal{X}$ , the norm induced by the inner product is

$$\|x\| \triangleq \sqrt{\langle x, x \rangle}$$

Let  $\mathcal{X}$  be an inner product space.

**Lemma.** (Cauchy-Schwartz Inequality) For all  $x, y \in \mathcal{X}$ ,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

with equality if and only if  $x = \lambda y$  or  $y = 0$ .

**Proof.** If  $y = 0$ , the result is clear. If  $y \neq 0$ ,

$$0 \leq \langle x - \lambda y, x - \lambda y \rangle = \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle$$

Set  $\lambda = \langle x, y \rangle / \langle y, y \rangle$ , then

$$0 \leq \langle x, x \rangle - \frac{\langle x, y \rangle^2}{\langle y, y \rangle}$$

□

**Theorem.** An inner product space equipped with the induced norm is a normed linear space.

**Proof.**

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2\end{aligned}$$

□

## Examples

---

**Example.**  $\mathbb{R}^n$ ,  $\langle x, y \rangle = x^T y$ ,  $\|x\| = \text{Euclidean norm}$

**Example.**  $\ell_2$ ,  $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ ,  $\|x\| = \ell_2\text{-norm}$

**Example.**  $L_2[a, b]$ ,  $\langle x, y \rangle = \int_a^b x(t)y(t) dt$ ,  $\|x\| = L_2\text{-norm}$

**Definition.** A complete inner product space is called a **Hilbert space**.

$\mathbb{R}^n$ ,  $\ell_2$ , and  $L_2[a, b]$  are all Hilbert spaces

Let  $\mathcal{X}$  be a Hilbert space, and  $\mathcal{C} \subset \mathcal{X}$  a **closed** and non-empty convex set. Fix the vector  $x \in \mathcal{X}$ .

$$\begin{array}{ll} \text{minimize} & \|z - x\| \\ \text{subject to} & z \in \mathcal{C} \end{array}$$

**Theorem.** For every  $x \in \mathcal{X}$ , the optimization problem has a unique global minimum  $x^*$  called the **projection** of  $x$  onto  $\mathcal{C}$ . A vector  $x' \in \mathcal{C}$  is equal to  $x^*$  if and only if

$$\langle x - x', z - x' \rangle \leq 0, \quad \forall z \in \mathcal{C}$$

## Projection Theorem: Proof

---

To prove existence, let  $z_i \in \mathcal{C}$  be a sequence with

$$\|z_i - x\| \rightarrow \delta \triangleq \inf_{z \in \mathcal{C}} \|z - x\|$$

Note that for all  $w, v \in \mathcal{X}$ ,

$$2\|w\|^2 + 2\|v\|^2 = \|w + v\|^2 + \|w - v\|^2 \quad (\text{Parallelogram law})$$

Then,

$$\|z_i - z_j\|^2 = 2\|z_i - x\|^2 + 2\|z_j - x\|^2 - 4 \left\| x - \frac{z_i + z_j}{2} \right\|^2$$

Since  $\mathcal{C}$  is convex,

$$\left\| x - \frac{z_i + z_j}{2} \right\| \geq \delta$$

Thus

$$\|z_i - z_j\|^2 \leq 2\|z_i - x\|^2 + 2\|z_j - x\|^2 - 4\delta^2 \rightarrow 0$$

Then,  $\{z_i\}$  is a Cauchy sequence, thus  $z_i \rightarrow x^* \in \mathcal{C}$  and  $\|x^* - x\| = \delta$  (continuity).

To prove uniqueness, let  $\tilde{x}^* \in \mathcal{C}$  be a point with  $\|\tilde{x}^* - x\| = \delta$ . Define

$$z_i = \begin{cases} x^* & i \text{ odd} \\ \tilde{x}^* & i \text{ even} \end{cases}$$

Clear  $\|z_i - x\| \rightarrow \delta$ , so by the same argument as before,  $\{z_i\}$  is a Cauchy sequence and convergent. Then,  $\tilde{x}^* = x^*$ .

## Projection Theorem: Proof

---

We wish to show that

$$\langle x - x^*, z - x^* \rangle \leq 0, \quad \forall z \in \mathcal{C}$$

Suppose there is some  $z_1$  with

$$\langle x - x^*, z_1 - x^* \rangle = \epsilon > 0$$

Define

$$z(\alpha) \triangleq (1 - \alpha)x^* + \alpha z_1, \quad 0 \leq \alpha \leq 1$$

Then,

$$\|x - z(\alpha)\|^2 = (1 - \alpha)^2 \|x - x^*\|^2 + 2\alpha(1 - \alpha) \langle x - x^*, x - z_1 \rangle + \alpha^2 \|x - z_1\|^2$$

This is a differentiable function of  $\alpha$ , and

$$\left. \frac{d}{d\alpha} \|x - z(\alpha)\|^2 \right|_{\alpha=0} = -2 \langle x - x^*, z_1 - x^* \rangle = -2\epsilon < 0$$

This contradicts the optimality of  $x^*$ .

Conversely, suppose that there exists some  $x' \in \mathcal{C}$  with

$$\langle x - x', z - x' \rangle \leq 0, \quad \forall z \in \mathcal{C}$$

Then, if  $z \in \mathcal{C}$ ,  $z \neq x'$ ,

$$\begin{aligned} \|x - z\|^2 &= \|x - x' + x' - z\|^2 \\ &= \|x - x'\|^2 + 2\langle x - x', x' - z \rangle + \|x' - z\|^2 \\ &> \|x - x'\|^2 \end{aligned}$$

Thus,  $x'$  is the unique optimizer.

## Linear Functionals

---

**Definition.** If  $\mathcal{X}$  is a vector space, a **linear functional** is a functional  $f : \mathcal{X} \rightarrow \mathbb{R}$  such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

**Example.** On  $\mathbb{R}^n$ , for every  $y \in \mathbb{R}^n$ , the function

$$f(x) = y^\top x$$

is a linear functional. Moreover, all linear functionals are of this form.

**Example.** On  $C[0, 1]$ , the functional

$$f(x) = x(1/2)$$

is a linear functional.

**Example.** On  $L_2[0, 1]$ , for every  $y \in L_2[0, 1]$ , the functional

$$f(x) = \int_0^1 x(t)y(t) dt$$

is a linear functional.

B9824 Foundations of Optimization

## Lecture 12: Vector Space Optimization II

Fall 2008

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### Outline

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1. Linear functionals
2. Dual spaces
3. Hahn-Banach Theorem
4. Minimum norm duality
5. Geometric Hahn-Banach Theorem

## Linear Functionals

---

**Definition.** If  $\mathcal{X}$  is a vector space, a **linear functional** is a functional  $f : \mathcal{X} \rightarrow \mathbb{R}$  such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

**Example.** On  $\mathbb{R}^n$ , for every  $y \in \mathbb{R}^n$ , the function

$$f(x) = y^\top x$$

is a linear functional. Moreover, all linear functionals are of this form.

**Example.** On  $C[0, 1]$ , the functional

$$f(x) = x(1/2)$$

is a linear functional.

**Example.** On  $L_2[0, 1]$ , for every  $y \in L_2[0, 1]$ , the functional

$$f(x) = \int_0^1 x(t)y(t) dt$$

is a linear functional.

## Linear Functionals

---

Let  $\mathcal{X}$  be a normed linear space.

**Theorem.** If a linear functional is continuous at a point in  $\mathcal{X}$ , it is continuous over all of  $\mathcal{X}$ .

**Proof.** Suppose  $f$  is linear and continuous at  $y$ , and  $x_n \rightarrow x$ . Then,  $x_n + y - x \rightarrow y$ . Thus,

$$|f(x_n) - f(x)| = |f(x_n + y - x) - f(y)| \rightarrow 0$$

by the linearity and continuity at  $y$  of  $f$ . □

Most commonly, we check that a linear functional is continuous just at 0.

# A Discontinuous Linear Functional

---

## Example.

$\mathcal{X} = \{x \in \mathbb{R}^\infty : x \text{ has finitely many non-zero components}\}$

$$\|x\| = \max_i |x_i|$$

Consider the linear functional

$$f(x) = \sum_{k=1}^{\infty} kx_k$$

Define  $x_k$  to have  $1/\sqrt{k}$  in the  $k$ th component, zero everywhere else.

$$\begin{aligned}\|x_k - \mathbf{0}\| &= 1/\sqrt{k} \rightarrow 0 \\ |f(x_k) - f(\mathbf{0})| &= \sqrt{k} \rightarrow \infty \neq f(\mathbf{0})\end{aligned}$$

## Linear Functionals

---

Let  $\mathcal{X}$  be a normed linear space.

**Definition.** A linear functional  $f$  is **bounded** if there is a constant  $M$  such that

$$|f(x)| \leq M\|x\|, \quad \forall x \in \mathcal{X}$$

**Theorem.** A linear functional is continuous if and only if it is bounded.

**Proof.** Suppose  $f$  is a bounded linear functional, with  $|f(x)| \leq M\|x\|$ . Then, if  $x_n \rightarrow \mathbf{0}$ ,  $|f(x_n)| \leq M\|x_n\| \rightarrow 0$ . Thus,  $f$  is continuous.

Conversely, assume that  $f$  is continuous. Then, there exists a  $\delta > 0$  such that  $|f(x)| < 1$  for  $\|x\| \leq \delta$ . Thus, if  $x \neq 0$ ,

$$|f(x)| = \left| f\left(\frac{\delta x}{\|x\|}\right) \right| \frac{\|x\|}{\delta} < \frac{\|x\|}{\delta},$$

and  $1/\delta$  is a bound for  $f$ . □

## Dual Spaces

---

Let  $\mathcal{X}$  be a normed linear space. We define the **normed dual space**  $\mathcal{X}^*$  to be the space of bounded linear functionals on  $\mathcal{X}$ , equipped with

- addition:  $(f_1 + f_2)(x) \triangleq f_1(x) + f_2(x)$
- scalar multiplication:  $(\alpha f)(x) \triangleq \alpha f(x)$
- zero element:  $\mathbf{0}(x) \triangleq 0$
- norm:

$$\begin{aligned}\|f\|_* &\triangleq \inf \{M : |f(x)| \leq M\|x\|, \forall x \in \mathcal{X}\} \\ &= \sup_{x \neq \mathbf{0}} \frac{|f(x)|}{\|x\|} = \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\|=1} |f(x)|\end{aligned}$$

Given a bounded linear functional  $x^* \in \mathcal{X}^*$ , we will abuse notation to write

$$\langle x, x^* \rangle \triangleq x^*(x)$$

## Dual Spaces

---

**Theorem.** If  $\mathcal{X}$  is a normed linear space, the dual space  $\mathcal{X}^*$  is a Banach space.

**Proof.** Clearly  $\mathcal{X}^*$  is a normed linear space. We need to show that it is complete.

Given a Cauchy sequence  $\{x_n^*\}$  and  $x \in \mathcal{X}$ , note that

$$|x_n^*(x) - x_m^*(x)| \leq \|x_n^* - x_m^*\| \|x\|$$

Then,  $\{x_n^*(x)\}$  is a Cauchy sequence. Define  $x^*$  pointwise by  $x^*(x) = \lim x_n^*(x)$ . It is easy to verify that  $x^*$  is a linear operator, it is bounded, and  $\|x^* - x_n^*\| \rightarrow 0$ .  $\square$

## Some Common Duals

---

**Example.**  $\mathcal{X} = \mathbb{R}^n$ ,  $\|x\| = \|x\|_2 = \sqrt{x^\top x}$   
 $\Rightarrow \mathcal{X}^* = \mathbb{R}^n$ ,  $\|x^*\|_* = \|x\|_2$

This space is **self-dual**.

**Theorem.** (Riesz-Fréchet) If  $\mathcal{X}$  is a Hilbert space, then  $\mathcal{X}^* = \mathcal{X}$ .

**Example.**  $\mathcal{X} = \ell_p$ ,  $1 \leq p < \infty$   
 $\Rightarrow \mathcal{X}^* = \ell_q$ , where  $1/p + 1/q = 1$  (if  $p = 1$ ,  $q = \infty$ )

**Note:** The dual to  $\ell_\infty$  is not  $\ell_1$ !

**Example.**  $\mathcal{X} = L_p[a, b]$ ,  $1 \leq p < \infty$   
 $\Rightarrow \mathcal{X}^* = L_q[a, b]$ , where  $1/p + 1/q = 1$  (if  $p = 1$ ,  $q = \infty$ )

**Note:** The dual to  $L_\infty[a, b]$  is not  $L_1[a, b]$ !

## Functions of Bounded Variation

---

**Definition.** Given a function  $x : [a, b] \rightarrow \mathbb{R}$ , define the **total variation** to be

$$\text{TV}(x) \triangleq \sup \sum_{i=1}^n |x(t_i) - x(t_{i-1})|$$

where the supremum is taken over all partitions

$$a = t_0 \leq t_1 \leq \dots \leq t_n = b$$

of  $[a, b]$ . It is often written as

$$\text{TV}(x) = \int_a^b |dx(t)|$$

**Definition.**  $BV[a, b]$  is the space of functions on  $[a, b]$  of bounded total variation with norm

$$\|x\| = |x(a)| + \text{TV}(x)$$

**Theorem.** (Riesz Representation Theorem) Suppose that  $f$  is a bounded linear functional on  $C[a, b]$ . Then, there is a function  $v \in BV[a, b]$  such that for all  $x \in C[a, b]$

$$f(x) = \int_a^b x(t) dv(t)$$

**Note:** The representation  $v$  of a linear functional  $f$  is not unique. To remove this ambiguity, define  $NBV[a, b]$  to be the set of functions  $x \in BV[a, b]$  with  $x(a) = 0$  that are right continuous on  $(a, b)$ . Then,  $C[a, b]^* = NBV[a, b]$ .

## Hahn-Banach Theorem: Extension Form

---

**Definition.** A **sublinear functional** is a map  $p : \mathcal{X} \rightarrow \mathbb{R}$  such that

1.  $p(x + y) \leq p(x) + p(y), \quad \forall x, y \in \mathcal{X}$
2.  $p(\alpha x) = \alpha p(x), \quad \forall x \in \mathcal{X}, \alpha \geq 0$

**Theorem.** (Hahn-Banach) Let  $\mathcal{X}$  be a normed linear space and  $p$  a continuous, sublinear functional on  $\mathcal{X}$ . Suppose  $\mathcal{M} \subset \mathcal{X}$  is a subspace, and  $f$  is a linear functional on  $\mathcal{M}$ , with

$$f(m) \leq p(m), \quad \forall m \in \mathcal{M}$$

Then, there is an extension of  $F$  of  $f$  from  $\mathcal{M}$  to  $\mathcal{X}$  such that

$$F(x) \leq p(x), \quad \forall x \in \mathcal{X}$$

## Hahn-Banach Theorem: Proof Sketch

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We describe the “induction” step: suppose  $y \in \mathcal{X} \setminus \mathcal{M}$ , we will extend  $f$  to the subspace

$$\mathcal{M}' = \{m + \alpha y : m \in \mathcal{M}, \alpha \in \mathbb{R}\}$$

Given  $m_1, m_2 \in \mathcal{M}$ ,

$$f(m_1) + f(m_2) = f(m_1 + m_2) \leq p(m_1 + m_2) \leq p(m_1 - y) + p(m_2 + y)$$

$$\Rightarrow f(m_1) - p(m_1 - y) \leq p(m_2 + y) - f(m_2)$$

$$\Rightarrow c \triangleq \sup_{m \in \mathcal{M}} f(m) - p(m - y) \leq \inf_{m \in \mathcal{M}} p(m + y) - f(m)$$

## Hahn-Banach Theorem: Proof Sketch

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If  $x \in \mathcal{M}'$ , the  $x$  can be uniquely written as  $m + \alpha y$ , and we define the extension

$$g(x) = f(m) + \alpha c$$

Clearly  $g$  is linear. We would like to show that

$$g(m + \alpha y) \leq p(m + \alpha y)$$

If  $\alpha > 0$ ,

$$\begin{aligned} f(m) + \alpha c &= \alpha \left[ c + f\left(\frac{m}{\alpha}\right) \right] \leq \alpha \left[ p\left(\frac{m}{\alpha} + y\right) - f\left(\frac{m}{\alpha}\right) + f\left(\frac{m}{\alpha}\right) \right] \\ &= p(m + \alpha y) \end{aligned}$$

$\alpha < 0$  is handled similarly.

## Hahn-Banach Theorem

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**Corollary.** Let  $f$  be a bounded linear functional on a subspace  $\mathcal{M}$  of a normed linear space  $\mathcal{X}$ . There is an extension  $F$  defined on  $\mathcal{X}$  with  $\|F\|_* = \|f\|_{\mathcal{M},*}$ .

**Proof.** Take  $p(x) = \|f\|_{\mathcal{M},*}\|x\|$ . □

**Corollary.** Let  $x$  be an element of a normed linear space  $\mathcal{X}$ . There exists a bounded non-zero linear functional  $F$  with  $F(x) = \|F\|_*\|x\|$ .

**Proof.** Suppose  $x \neq 0$ . On the subspace defined by  $x$ , define  $f(\alpha x) = \alpha\|x\|$ . Clearly  $\|f\| = 1$ . Extend  $f$  to the entire space.

If  $x = 0$ , any non-zero bounded linear functional will do, and we have just proved that one exists. □

## Optimization Interpretation

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Given a normed linear space  $\mathcal{X}$ , a subspace  $\mathcal{M}$ , and a bounded linear functional  $f$  on  $\mathcal{M}$ , consider the optimization problem

$$\begin{aligned} & \text{minimize} && \|x^*\|_* \\ & \text{subject to} && \langle x, x^* \rangle = f(x), \quad \forall x \in \mathcal{M} \\ & && x^* \in \mathcal{X}^* \end{aligned}$$

The Hahn-Banach theorem implies that a global optimum exists, and that the optimal value is  $\|f\|_{\mathcal{M},*}$ .

**Definition.** Vectors  $x \in \mathcal{X}$  and  $x^* \in \mathcal{X}^*$  are **aligned** if

$$\langle x, x^* \rangle = \|x^*\|_* \|x\|$$

**Definition.** Vectors  $x \in \mathcal{X}$  and  $x^* \in \mathcal{X}^*$  are **orthogonal** if

$$\langle x, x^* \rangle = 0$$

**Definition.** The **orthogonal complement** of a set  $\mathcal{M} \subset \mathcal{X}$  is

$$\mathcal{M}^\perp = \{x^* \in \mathcal{X}^* : \langle x, x^* \rangle = 0\}$$

## Minimum Norm Duality

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**Theorem.** Suppose  $\mathcal{X}$  is a normed linear space,  $\mathcal{M} \subset \mathcal{X}$  is a subspace, and  $x \in \mathcal{X}$ . Then,

$$\inf_{m \in \mathcal{M}} \|x - m\| = \max_{\substack{\|x^*\|_* \leq 1 \\ x^* \in \mathcal{M}^\perp}} \langle x, x^* \rangle$$

where the maximum is achieved for some  $x_0^* \in \mathcal{M}^\perp$ .

Further, the minimum is achieved by some  $m_0 \in \mathcal{M}$  if and only if  $x_0^*$  is aligned with  $x - m_0$ .

## Minimum Norm Duality: Proof

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Set  $d \triangleq \inf_{m \in \mathcal{M}} \|x - m\|$ .

For any  $\epsilon > 0$ , suppose  $m_\epsilon \in \mathcal{M}$  with  $\|x - m_\epsilon\| \leq d + \epsilon$ . If  $x^* \in \mathcal{X}^*$  is feasible for the dual,

$$\langle x, x^* \rangle = \langle x - m_\epsilon, x^* \rangle \leq \|x^*\|_* \|x - m_\epsilon\| \leq d + \epsilon$$

Since  $\epsilon$  was arbitrary,  $\langle x, x^* \rangle \leq d$ .

Let  $\mathcal{N}$  be the subspace spanned by  $\mathcal{M}$  and  $x$ . Define  $f$  on  $\mathcal{N}$  by

$$f(\alpha x + m) = \alpha d, \quad \forall \alpha \in \mathbb{R}, m \in \mathcal{M}$$

Then,

$$\|f\|_{\mathcal{N},*} = \sup \frac{|\alpha|d}{\|\alpha x + m\|} = \sup \frac{|\alpha|d}{|\alpha| \|x + m/\alpha\|} = \frac{d}{\inf \|x + m/\alpha\|} = 1$$

By H-B, there exists  $x_0^* \in \mathcal{X}$  that is an extension of  $f$ , and  $\|x_0^*\|_* = 1$ . Clearly  $x_0^* \in \mathcal{M}^\perp$  and  $\langle x, x_0^* \rangle = d$ , so  $x_0^*$  is optimal for the dual.

## Minimum Norm Duality: Proof

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Suppose  $m_0$  is a primal optimal, and  $x_0^*$  is dual optimal. Then,

$$\langle x - m_0, x_0^* \rangle = \langle x, x_0^* \rangle = d = \|x - m_0\|$$

Thus,  $x - m_0$  and  $x_0^*$  are aligned.

Conversely, assume that  $x_0^*$  is dual optimal,  $m_0$  is feasible for the primal, and  $x - m_0$  and  $x_0^*$  are aligned. For any  $m \in \mathcal{M}$ ,

$$\langle x, x_0^* \rangle = \langle x - m, x_0^* \rangle \leq \|x - m\|$$

But,

$$\langle x, x_0^* \rangle = \langle x - m_0, x_0^* \rangle = \|x - m_0\|$$

Thus,  $m_0$  is primal optimal.

**Theorem.** Suppose  $\mathcal{X}$  is a normed linear space,  $\mathcal{M} \subset \mathcal{X}$  is a subspace, and  $x^* \in \mathcal{X}^*$ . Then,

$$\min_{m^* \in \mathcal{M}^\perp} \|x^* - m^*\|_* = \sup_{\substack{\|x\| \leq 1 \\ x \in \mathcal{M}}} \langle x, x^* \rangle$$

where the minimum is achieved for some  $m_0^* \in \mathcal{M}^\perp$ .

Further, the maximum is achieved by some  $x_0 \in \mathcal{M}$  if and only if  $x_0$  is aligned with  $x^* - m_0^*$ .

## Example: Rocket Control

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We would like to optimize the fuel consumption of a rocket:

- $x(t)$  = altitude of a rocket at time  $t$
- $u(t)$  = thrust applied at time  $t$
- system dynamics:

$$\ddot{x}(t) = u(t) - 1, \quad x(0) = \dot{x}(0) = 0$$

- constraint: achieve a certain altitude by time  $T$  (free variable),

$$x(T) = 1$$

- fuel consumption =  $\int_0^T |u(t)| dt$

## Example: Rocket Control

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For now, fix  $T$  (we will optimize over  $T$  later).

By integration by parts, the dynamics be written as

$$x(T) = \int_0^T (T-t)u(t) dt - \frac{1}{2}T^2$$

$$\begin{aligned} &\text{minimize} && \int_0^T |u(t)| dt \\ &\text{subject to} && \int_0^T (T-t)u(t) dt = \frac{1}{2}T^2 + 1 \end{aligned}$$

We would like to formulate this as a minimum norm problem in a dual space.

## Example: Rocket Control

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$$\begin{aligned} \Rightarrow & \text{minimize} && \int_0^T |dv(t)| \\ & \text{subject to} && \int_0^T (T-t) dv(t) = \frac{1}{2}T^2 + 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow & \text{minimize} && \|v\| \\ & \text{subject to} && \langle T-t, v \rangle = \frac{1}{2}T^2 + 1 \\ & && v \in \text{NBV}[0, T] \end{aligned}$$

Let  $\bar{v} \in \text{NBV}[0, T]$  be some feasible vector, and let  $\mathcal{M} \subset C[0, T]$  be the subspace spanned by  $T-t$ . Then,

$$\min_{\langle T-t, v \rangle = \frac{1}{2}T^2 + 1} \|v\| = \min_{v' \in \mathcal{M}^\perp} \|\bar{v} - v'\| = \sup_{\substack{x \in \mathcal{M} \\ \|x\| \leq 1}} \langle x, \bar{v} \rangle$$

## Example: Rocket Control

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$$\sup_{\substack{x \in \mathcal{M} \\ \|x\| \leq 1}} \langle x, \bar{v} \rangle = \sup_{\substack{\alpha \in \mathbb{R} \\ \|\alpha(T-t)\| \leq 1}} \langle \alpha(T-t), \bar{v} \rangle = \sup_{\substack{\alpha \in \mathbb{R} \\ \|\alpha(T-t)\| \leq 1}} \alpha \left( \frac{1}{2} T^2 + 1 \right)$$

However,

$$\|\alpha(T-t)\| = \max_{t \in [0, T]} |\alpha(T-t)| = |\alpha| T$$

Then, the maximum is achieved when  $\alpha = 1/T$ , thus

$$\min_{\langle T-t, v \rangle = \frac{1}{2} T^2 + 1} \|v\| = \frac{1}{2} T + 1/T$$

The optimal  $v$  must be aligned with  $\alpha(T-t)$ , therefore it is a step function at  $t = 0$  (impulse control). Now, optimizing over  $T$ , we obtain

$$T^* = \sqrt{2}, \quad v^*(t) = \begin{cases} 0 & \text{if } t = 0 \\ \sqrt{2} & \text{if } t \in (0, \sqrt{2}] \end{cases}$$

## Hyperplanes

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**Definition.** A **hyperplane**  $H$  in a normed linear space  $\mathcal{X}$  is maximal proper affine set. In other words,  $H \neq \mathcal{X}$ , and if  $V$  is an affine set with  $H \subset V$ , then  $V = \mathcal{X}$ .

**Theorem.** A set  $H$  is a hyperplane if and only if it is of the form

$$\{x \in \mathcal{X} : f(x) = c\}$$

where  $f$  is a non-zero linear functional and  $c$  is a scalar.

**Theorem.** The hyperplane  $H = \{x : f(x) = c\}$  is closed for all scalars  $c$  if and only if  $f$  is continuous.

**Definition.** A **halfspace** is a set of the form  $\{x : f(x) \leq c\}$ . It is closed if  $f$  is continuous.

**Definition.** Suppose  $\mathcal{K}$  is a convex subset of a normed linear space  $\mathcal{X}$  and  $0 \in \text{int } \mathcal{K}$ . The **Minkowski functional**  $p : \mathcal{X} \rightarrow \mathbb{R}$  is defined by

$$p(x) = \inf \{r \in \mathbb{R} : x/r \in \mathcal{K}, r > 0\}$$

**Lemma.** The Minkowski functional satisfies:

1.  $0 \leq p(x) < \infty$
2.  $p(\alpha x) = \alpha p(x)$ , for  $\alpha > 0$
3.  $p(x_1 + x_2) \leq p(x_1) + p(x_2)$
4.  $p$  is continuous
5.  $\text{cl } \mathcal{K} = \{x : p(x) \leq 1\}$ ,  $\text{int } \mathcal{K} = \{x : p(x) < 1\}$

## Geometric Hahn-Banach Theorem

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**Theorem.** Let  $\mathcal{K}$  be a convex set with a non-empty interior in a normed linear space  $\mathcal{X}$ . Suppose  $\mathcal{V} \subset \mathcal{X}$  is an affine set containing no interior points of  $\mathcal{K}$ . Then, there exists a closed hyperplane  $H$  containing  $\mathcal{V}$  but no interior points of  $\mathcal{K}$ . In other words, there exists  $x^* \in \mathcal{X}^*$  and  $c \in \mathbb{R}$  such that

$$\begin{aligned} \langle x, x^* \rangle &= c, & \forall x \in \mathcal{V} \\ \langle x, x^* \rangle &< c, & \forall x \in \text{int } \mathcal{K} \end{aligned}$$

## Geometric Hahn-Banach Theorem: Proof

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WLOG, assume that  $0 \in \text{int } \mathcal{K}$ . Let  $\mathcal{M}$  be the subspace of  $\mathcal{X}$  generated by  $\mathcal{V}$ . Since  $\mathcal{V}$  is a hyperplane of  $\mathcal{M}$  which does not contain  $0$ , there exists a functional  $f$  on  $\mathcal{M}$  with

$$\mathcal{V} = \{x \in \mathcal{M} : f(x) = 1\}$$

Since  $\mathcal{V}$  contains no interior points of  $\mathcal{K}$ ,

$$f(v) = 1 \leq p(v), \quad \forall v \in \mathcal{V}$$

By homogeneity,

$$f(\alpha v) = \alpha \leq p(\alpha v), \quad \forall v \in \mathcal{V}, \alpha > 0$$

$$f(\alpha v) \leq 0 \leq p(\alpha v), \quad \forall v \in \mathcal{V}, \alpha < 0$$

Thus,

$$f(x) \leq p(x), \quad \forall x \in \mathcal{M}$$

## Geometric Hahn-Banach Theorem: Proof

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By the Hahn-Banach Theorem, we can extend  $f$  to a functional  $F$  on  $\mathcal{X}$  with

$$F(x) \leq p(x), \quad \forall x \in \mathcal{X}$$

Since  $p$  is continuous, so is  $F$ , and  $F(x) < 1$  for all  $x \in \text{int } \mathcal{K}$ . Then, the desired hyperplane is

$$H = \{x \in \mathcal{X} : F(x) = 1\}$$